Fixing the Loose Brake: Exponential-Tailed Stopping Time in Best Arm Identification

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Abstract

The best arm identification problem requires identifying the best alternative (i.e., arm) in active experimentation using the smallest number of experiments (i.e., arm pulls), which is crucial for cost-efficient and timely decision-making processes. In the fixed confidence setting, an algorithm must stop datadependently and return the estimated best arm with a correctness guarantee. Since this stopping time is random, we desire its distribution to have light tails. Unfortunately, many existing studies focus on high probability or in expectation bounds on the stopping time, which allow heavy tails and, for high probability bounds, even not stopping at all. We first prove that this never-stopping event can indeed happen for some popular algorithms. Motivated by this, we propose algorithms that provably enjoy an exponential-tailed stopping time, which improves upon the polynomial tail bound reported by Kalyanakrishnan et al. (2012). The first algorithm is based on a fixed budget algorithm called Sequential Halving along with a doubling trick. The second algorithm is a meta algorithm that takes in any fixed confidence algorithm with a high probability stopping guarantee and turns it into one that enjoys an exponential-tailed stopping time. Our results imply that there is much more to be desired for contemporary fixed confidence algorithms.

1 Introduction

The multi-armed bandit model serves as a fundamental framework for many real-world sequential decision-

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making applications, such as clinical trials, online advertising, and web content optimization (Villar et al., 2015; Schwartz et al., 2017). At its core, the learner sequentially and strategically collects data, typically in the form of rewards or outcomes, from different alternatives (i.e., arms) to make critical decisions. The problem of best arm identification aims to identify the arm with the highest mean reward, either with a pre-specified confidence level or within a pre-specified sampling budget, known as the fixed confidence setting and fixed budget setting, respectively.

In the fixed confidence setting, one of the most popular applications is A/B/../K testing that many firms use to identify the most effective recommendation or treatment among several options. In practice, the randomization units (treatments or arm pulls) are randomly assigned to either the treatment or the control group, and the experimenter observes the outcomes until a decision can be made regarding whether one treatment is more effective than the others, with a pre-specified confidence level, such as 95%.

In the more advanced fixed confidence best arm identification setting, arm pulls are assigned adaptively based on the outcomes of previous pulls. The advantage of adaptive assignment is that it often results in significantly lower sample complexity compared to non-adaptive assignment. Therefore, many works have proposed algorithms and proved bounds on how many samples they use until stopping (i.e., sample complexity) either with high probability or in expectation (Even-Dar et al., 2006; Karnin et al., 2013b; Jamieson et al., 2014). However, most existing sample complexity guarantees do not sufficiently describe the behavior of the stopping time. For instance, an algorithm might be guaranteed to stop before T^* samples with probability at least $1 - \delta$, but with probability up to δ , the algorithm may never stop or stop only after a very long time, much larger than T^* . Algorithms with expected sample complexity guarantees will stop, but the tail of the distribution of the stopping time can be very thick. Thus, the realized stopping time can

be significantly larger than the expected one, which is undesirable. These issues have been largely overlooked in the literature.

To address this gap, we push the limits of attainable guarantees for the tail decaying rate of stopping time distribution of fixed confidence best arm identification algorithms. To the best of our knowledge, aside from LUCB by Kalyanakrishnan et al. (2012) with a polynomial-tailed stopping time bound, there has been no significant study on the tail of the stopping time distribution.¹ In particular, there are no results in the literature reporting an exponential tail bound for the stopping distribution except for the naive uniform sampling.

In this paper, we make several key contributions towards strengthening guarantees for the stopping time distribution, which we summarize as follows:

- Confirmation of an issue with existing guarantees: We confirm a critical issue in many existing fixed confidence best arm identification algorithms by showing that, despite enjoying highprobability guarantees, they will not stop at all with a constant probability.
- Fixed Confidence Doubling Sequential Halving (FC-DSH): We demonstrate that the Sequential Halving algorithm, when combined with the doubling trick and an appropriate stopping rule, achieves an exponential tail for the stopping time. To our knowledge, this is the first such guarantee in the literature for fixed confidence algorithms except for the naive uniform sampling.
- BrakeBooster: Motivated by our success with FC-DSH, we take a step further and propose a novel meta-algorithm approach. This approach takes any fixed confidence algorithm that meets mild conditions and and turns it into an algorithm with an exponential tail guarantee for the stopping time.

Our results convey a strong message to the community: despite decades of research, there are much more to be desired from the current state-of-the-art fixed confidence algorithms.

2 Problem Definition and Preliminaries

We consider the standard K-armed bandit setting, where a learner sequentially selects one of K arms where each arm $i \in [K]$ is associated with a reward distribution ν_i that is 1-sub-Gaussian (known) with mean μ_i (unknown). We assume that there exists a unique best arm, which is standard Audibert et al. (2010). Without loss of generality, we assume that the arms are ordered in decreasing order of their mean rewards, i.e., $\mu_1 > \mu_2 \geq \cdots \geq \mu_K$. At each time step t, the learner selects an arm $A_t \in [K] := \{1, 2, \dots, K\}$, then observes a reward $r_t \sim \nu_{A_t}$. We consider the fixed confidence setting, where the learner aims to identify the best arm with a pre-specified confidence level $\delta \in (0, 1)$ that is also called failure rate. The goal is to design an algorithm \mathcal{A} that includes a sampling rule choosing A_t , a stopping rule that determines when to stop, and a recommendation rule that outputs the estimated best arm $J(\mathcal{A})$ when stopping. We denote by $\tau(\mathcal{A})$ the (random) stopping time of the algorithm \mathcal{A} , which is the arm pulls that \mathcal{A} makes before the algorithm stops. We often omit the dependence on the algorithm \mathcal{A} and use τ and J when the algorithm being used is clear from context.

An algorithm for the fixed confidence setting is required to satisfied the following correctness result.

Definition 1 (δ -correct). A fixed confidence algorithm is said to be δ -correct if it takes δ as input and satisfies $\mathbb{P}(\tau < \infty, J \neq 1) \leq \delta$.

We call such an algorithm δ -correct. Note that the condition $\tau(\mathcal{A}) < \infty$ above is necessary since $J_{\tau}(\mathcal{A})$ is undefined otherwise.

Furthermore, we desire the algorithm to stop as early as possible in addition to being δ -correct. There are two criteria that have been popular in the literature: asymptotic expected sample complexity² and high probability sample complexity.

The asymptotic expected sample complexity (Chernoff, 1959; Garivier and Kaufmann, 2016; Shang et al., 2020; Qin et al., 2017) characterizes the asymptotic behavior of the stopping time as δ goes to 0 as follows.

Definition 2 (Asymptotic expected sample complexity). A fixed confidence algorithm is said to have an asymptotic expected (AE) sample complexity of T_{δ}^* if it satisfies

$$\liminf_{\delta \to 0} \frac{\mathbb{E}[\tau]}{\ln(1/\delta)} \le T_{\delta}^* ,$$

where τ depends on δ .

The optimal guarantee has been well-understood for exponential family reward models (Garivier and Kaufmann, 2016). For example, Track-and-Stop (Garivier and Kaufmann, 2016) achieves the optimal asymptotic

¹While AT-LUCB (Jun and Nowak, 2016) showed a tail bound that decays exponentially, the bound is exponential in \sqrt{t} rather than t, which does not fit Definition 1. Furthermore, the correctness is questionable; see Appendix .

 $^{^{2}}$ Recently, Jourdan and Degenne (2023) analyzed the nonasymptotic expected sample complexity.

expected sample complexity.

However, the AE sample complexity has two limitations. First, the AE sample complexity does not tell us anything about the tail of τ . In fact, τ can still be heavy-tailed as empirically observed by Jourdan et al. (2022, Figure 4, EB-TC) despite having a nearoptimal AE sample complexity. Second, the AE sample complexity hides potentially bad behaviors of the algorithm in the nonasymptotic regime or with moderately small δ . That is, the AE sample complexity hinges on the behavior of the algorithm when δ close enough to 0 in which case the algorithm would be running for a very long time. Indeed, one can observe that if $\mathbb{E}[\tau] \approx A \ln(1/\delta) + B$ for some A and B that are not dependent on δ , the value of $\liminf_{\delta \to 0} \frac{\mathbb{E}[\tau]}{\ln(1/\delta)}$ will be independent of B even if B is very large.

On the other hand, high probability sample complexity guarantee (Even-Dar et al., 2006; Karnin et al., 2013b; Jamieson et al., 2014; Jun et al., 2016; Tanczos et al., 2017), defined below, does not rely on the asymptotic behavior of the algorithm.

Definition 3 (High probability sample complexity). A fixed confidence algorithm is said to have a sample complexity of T^*_{δ} if it takes $\delta \in (0,1)$ as input and satisfies

$$\mathbb{P}(\tau \ge T^*_{\delta}) \le \delta$$

For example, Successive Elimination (Even-Dar et al., 2006) achieves a high probability sample complexity of $\tilde{\mathcal{O}}(H_1 \ln(1/\delta))$, where $H_1 := \sum_{i=2}^{K} \Delta_i^{-2}$ and $\tilde{\mathcal{O}}$ omits logarithmic factors except for $\ln(1/\delta)$. Despite being nonasymptotic, we find the high probability sample complexity weak and rather unnatural for the following reasons:

- First, we find it unnatural that δ , the failure rate regarding the *correctness* of the output J, is also the target failure rate with which we bound the *sample complexity*. In practice, one may desire to be loose on the correctness (large δ) yet want to ensure that the stopping time is small with *very high* confidence (small δ). We speculate that the high probability sample complexity is a mere byproduct of easy analysis.
- Second, more importantly, the guarantee above does not tell us about the shape of the tail of the stopping time. In particular, the high probability sample complexity guarantee does not exclude the possibility of never stopping even if the problem at hand is easy, which is a serious issue as it would imply that the practitioner will have to wait forever or forcefully stop the active experimentation procedure. It also follows that the expected stopping time E[τ] does not exist. See Figure 1



Figure 1: Historgram of stopping times of Successive Elimination Even-Dar et al. (2006) out of 1000 independent trials on three arms with mean rewards of $\{1.0, 0.9, 0.9\}$ and Gaussian noise. We forcefully terminated the runs that do not stop until 30,000 time steps. We have observed that all these runs have already eliminated the best arm, and thus we expect that many of them will never stop.

To further demonstrate the issue of having an extremely bad tail for the stopping time distribution, we provide a lower bound for the Successive Elimination algorithm (Even-Dar et al., 2006). Hereafter, all the proofs are deferred to the appendix unless stated otherwise.

Theorem 4. For Successive Elimination, there exists an instance with a unique best arm where the algorithm never stops with a constant probability.

We show that the same is true for KL-LUCB (Tanczos et al., 2017).

Theorem 5. For KL-LUCB, there exists an instance with a unique best arm where the algorithm never stops with a constant probability.

Remark 1. One way to stop the algorithm from infinitely running is to allow ε -slack in the stopping condition as done in, e.g., Kalyanakrishnan et al. (2012). However, one can also extend the arguments in Theorem 4 and 5 to show that the stopping time can be as large as $\Theta(K/\varepsilon^2)$ even if the guaranteed high probability sample complexity is significantly smaller than $\Theta(K/\varepsilon^2)$, and this gap can be made arbitrarily large. In this paper, we focus on the best arm identification to keep the discussion concise.

Meanwhile, the seminal work by Kalyanakrishnan et al. (2012) has proposed an algorithm called LUCB1 that satisfies the following polynomial tail guarantee on the stopping time, which is adapted for the best arm identification problem rather than the $\varepsilon\text{-optimal}$ arm identification.

Theorem 6 (Adapted from Kalyanakrishnan et al. (2012)). Let $T^* = \left[146H_1 \ln \left(\frac{H_1}{\delta} \right) \right]$. For every $T \ge T^*$, the probability that LUCB1 has not terminated after T samples is at most $\frac{4\delta}{T^2}$.

This makes us wonder if it is possible to achieve an exponentially-decaying tail bound for the stopping time τ , which we believe is important in practice. We formalize the desired property in the following definition where x = polylog(T) means $x \leq a \log^b(T) + c$ for some absolute and positive constants a, b, and c.

Definition 7 ((T_{δ}, κ) -exponential stopping tail). A fixed confidence algorithm is said to have a (T_{δ}, κ) -exponential stopping tail if there exists a time step T_{δ} and a problem-dependent constant $\kappa > 0$ (but not dependent on T) such that for all $T \geq T_{\delta}$,

$$\mathbb{P}\left(\tau \ge T\right) \le \exp\left(-\frac{T}{\kappa \cdot \operatorname{polylog}(T)}\right). \tag{1}$$

This property requires a tail bound for every large enough T, which reveals detailed information about the distribution function of the distribution of τ . Perhaps not surprising, this requirement is strictly stronger than the high probability sample complexity above and implies a few desirable properties regarding the stopping time τ as we summarize below.

Proposition 8. If a fixed confidence algorithm \mathcal{A} has (T_{δ}, κ) -exponential stopping tail, then the following holds true:

(i) $\mathbb{P}\left(\tau \geq T_{\delta} + \kappa \ln(1/\delta) \cdot \operatorname{polylog}(\kappa \ln(1/\delta))\right) \leq \delta.$ (ii) $\mathbb{E}[\tau] \leq T_{\delta} + \kappa \cdot \operatorname{polylog}(\kappa).$ (iii) $\mathbb{P}(\tau < \infty) = 1.$

Proof. For (i), find T that would make the RHS of (1) below δ . For (ii), use the identity $\mathbb{E}[X] = \sum_{x=0}^{\infty} \mathbb{P}(X > x)$. For (iii), use the Borel–Cantelli lemma.

While the guarantees above can be individually satisfied by a specific criterion such as high probability or expected sample complexity, Proposition 8 shows that enjoying an exponential tail guarantee implies all three desirable properties simultaneously.

One can show that the naive uniform sampling algorithm that chooses arm $A_t = 1 + ((t-1) \mod K)$ at time t with a suitable stopping condition results in (T_{δ}, κ) -exponential stopping tail with $T_{\delta} = \tilde{\Theta}(K\Delta_2^{-2}\ln(1/\delta))$ and $\kappa = \Theta(K\Delta_2^{-1})$. However, by Proposition 8, this guarantee is converted to a high probability sample complexity of $\tilde{\mathcal{O}}(K\Delta_2^{-2}\ln(1/\delta))$, which is much worse than $\tilde{\mathcal{O}}(H_1 \ln(1/\delta))$ achieved by Successive Elimination.

We thereby ask if it is possible to achieve $(\tilde{\Theta}(H_1 \ln(1/\delta)), H_1)$ -exponential stopping tail. We answer this question in the affirmative by proposing two algorithms in the following two sections.

3 Fixed Confidence Doubling Sequential Halving

In this section, we explore turning doubling Sequential Halving (DSH) algorithm, which was originally designed for the fixed budget setting, called FB-DSH, into a fixed confidence algorithm, called FC-DSH. We prove that a version of FC-DSH not only satisfies correctness guarantee but also achieves $(\tilde{\Theta}(H_2 \log(1/\delta)), \tilde{\mathcal{O}}(H_2))$ exponential stopping tail, where $H_2 = \max_{i=2}^{K} i \Delta_i^{-2}$. The FB-DSH introduced by Zhao et al. (2023) merges the Sequential Halving (SH) algorithm (Karnin et al., 2013b) with the doubling trick. In general, FC-DSH iteratively runs SH instances in phases, where each subsequent phase (m+1)-th is allocated a budget that is doubled from that of the preceding phase m-th. The initial budget T_1 is $|K \log_2(K)|$, and the subsequent budget is $T_m = 2^{m-1}T_1$, $\forall m \ge 2$. At the end of phase m, the *m*-th SH instance outputs arm J_m . FB-DSH is a fixed budget algorithm in which the algorithm is given a pre-determined budget and the algorithm terminates when the given budget is ran out. In contrast, FC-DSH is a fixed confidence algorithm in which the algorithm is given a fixed confidence level $\delta \in (0, 1)$ and has a stopping condition that depends on δ , and the algorithm terminates when the stopping condition is satisfied. In this paper, we present a version of FC-DSH with a stopping condition where the output arm J_m from m-th SH must have its lower confidence bound greater than the maximum upper confidence bounds of the remaining arms calculated at their elimination stages.

In more details, in phase m, when an arm i is eliminated at the stage ℓ_i , it has been sampled $N^{(m,\ell_i)}$ times and has received rewards $\left\{r_j^{(m,\ell_i)}\right\}_{j=1}^{N^{(m,\ell_i)}}$. Within this context, we define the empirical mean of arm i and its confidence width respectively as

$$\hat{\mu}_i^{(m)} := \frac{1}{N^{(m,\ell_i)}} \sum_{j=1}^{N^{(m,\ell_i)}} r_j^{(m,\ell_i)}$$

and

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$$p_i^{(m)} := \sqrt{\frac{2}{N^{(m,\ell_i)}} \log\left(\frac{6K \left\lceil \log_2(K) \right\rceil m^2}{\delta}\right)}.$$

Also, the upper and lower confidence bound of arm i

Algorithm 1 FC-DSH

Input: A set of K arms, δ $T_1 = \lceil K \log_2 K \rceil$ $T_m = T_1 2^{m-1}, \forall m \ge 2$ // the sampling budget assigned to phase m $L = \lceil \log_2(K) \rceil$ // the last stage in each phase **for** $m = 1, 2, \dots$ **do** // phase m Reset $\mathcal{A}_1 = \lceil K \rceil$

for $\ell = 1, \dots, L$ do

// stage
$$\ell$$

Sample each arm $i \in \mathcal{A}_{\ell}$ for $N^{(m,\ell)}$ times where

$$N^{(m,\ell)} = \left\lfloor \frac{T_m}{K2^{-\ell+1} \left\lceil \log_2(K) \right\rceil} \right\rfloor$$

Let $\mathcal{A}_{\ell+1}$ be the set of $\lceil \mathcal{A}_{\ell}/2 \rceil$ arms in \mathcal{A}_{ℓ} with the largest empirical means computed using samples from this stage only.

end for

Set J_m as the only arm in \mathcal{A}_L . if $L_{J_m}^{(m)} \ge \max_{i \ne J_m} U_i^{(m)}$ then Stop and output J_m . end if end for

are defined as,

 $U_i^{(m,\ell_i)} \coloneqq \hat{\mu}_i^{(m)} + b_i^{(m)} \text{ and } L_i^{(m)} \coloneqq \hat{\mu}_i^{(m)} - b_i^{(m)}.$

With these definitions in place, we define a stopping condition $L_{J_m}^{(m)} \geq \max_{i \neq J_m} U_i^{(m)}$. Intuitively, the stopping condition ensures that the selected arm J_m is statistically significantly better than all other remaining arms. Note that the pre-specified confidence level δ is an input to the confidence width and inherently affects the stopping condition. The full algorithm is presented in Algorithm 1.

The complexity of best-arm identification problems are often characterized by an instance-dependent quantity $H_2 = \max_{i=2}^{K} i\Delta_i^{-2}$. Audibert et al. (2010) show that H_2 is equivalent to H_1 up to a logarithmic factor,

$$H_2 \le H_1 \le \log(2K)H_2.$$

The following theorems show that DSH satisfies the correctness guarantee and enjoys an exponential stopping tail.

Theorem 9 (Correctness). *FC-DSH runs with confidence level* δ *is* δ *-correct.*

Theorem 10 (Exponential tail bound). *FC-DSH enjoys* $(\tilde{\Theta}(H_2 \log(1/\delta)), \tilde{\mathcal{O}}(H_2))$ -exponential stopping tail.

The proofs are deferred to Appendix B.1 and B.2 respectively.

The theorem above confirms our intuition that

there exists a fixed confidence algorithm achieving $(\tilde{\Theta}(H_2 \log(1/\delta)), \tilde{\mathcal{O}}(H_2))$ -exponential stopping tail. With our new insights, in the next section, we develop a novel meta algorithm that also enjoys exponential stopping tail.

4 BrakeBooster: A meta algorithm approach

In this section, we propose a novel algorithm called BrakeBooster. This is a meta algorithm in the sense that it takes any fixed confidence best arm identification algorithm (denoted by \mathcal{A}) equipped with the standard guarantees on the correctness and stopping time as an input and convert it into one that enjoys an exponential stopping tail.

We denote by $\mathcal{A}(\delta)$ as algorithm \mathcal{A} run with a target failure rate of δ . We assume that $\mathcal{A}(\delta)$ is δ -correct (Definition 1) and has a sample complexity guarantee of $T^*_{\delta}(\mathcal{A})$ (Definition 3). Note that BrakeBooster will not require T^*_{δ} as input, but only the existence.

Given such an algorithm \mathcal{A} , we are ready to describe BrakeBooster whose full pseudocode can be found in Algorithm 2. The key idea of BrakeBooster is to repeatedly invoke our key subroutine BudgetedIdentification (Algorithm 3) with increasing trial count $L_{r,c}$ and budget $T_{r,c}$ until a stopping criterion is met where (r, c) is an index of each invocation (called *stage*) with $r \in \mathbb{N}_+$ and $c \in [r]$. We defer the explanation on how we schedule $L_{r,c}$ and $T_{r,c}$ to the next paragraph. In stage (r,c), we are given the number $L = L_{r,c}$ of trials, the sampling budget $T = T_{r,c}$, and the base failure rate δ_0 , and run $\mathcal{A}(\delta_0)$ repeatedly L times with a sampling budget of T. Since \mathcal{A} itself may not stop before T time steps, we ensure the budget constraint by forcing \mathcal{A} to stop (*forced-termination*) when it does not stop by itself (*self-termination*) after exhausting T samples. For each trial $\ell \in [L]$, we collect the returned arm index \hat{J}_{ℓ} , which is set to 0 if \mathcal{A} was forced-terminated. If at least a half of the trials were forced-terminated, then we have failed - we return 0 from BudgetedIdentification. Otherwise, we declare success and return the majority vote over $\{\hat{J}_{\ell} : \ell \in [L], \hat{J}_{\ell} \neq 0\}$ (i.e., majority votes over nonzero votes). In the latter case, Brake-Booster stops and outputs the majority vote as the final output $J_{\tau}(\mathcal{M})$. In the former case, we continue to the next stage with trial count $L_{r',c'}$ and $T_{r',c'}$ where (r', c') = (r, c+1) if $c \le r-1$ and (r', c') = (r+1, 1)if c = r.

The key in our algorithm is the particular scheduling of $L_{r,c}$ and $T_{r,c}$, which is inspired by Li et al. (2018) and can be viewed as a two-dimensional doubling trick. We visualize the schedule in Figure 2. Each dot in the figure

Algorithm 2 BrakeBooster
Input: base trial count L_1 , base budget: T_1 , algo-
rithm \mathcal{A} , base failure rate δ_0
for $r = 1, 2,$ do
for $c = 1, 2,, r$ do
$L_{r,c} := r \cdot 2^{r-c} L_1, \ T_{r,c} := 2^{c-1} T_1$
$J_{r,c} = \text{BudgetedIdentification}(\mathcal{A}, L_{r,c}, T_{r,c}, \delta_0)$
if $J_{r,c} \neq 0$ then
$\mathbf{return} \ J_{r,c}$
end if
end for
end for

Algorithm 3 BudgetedIdentification

return arg $\max_{i \in [K]} v_i$

end if

Input: algorithm \mathcal{A} , the number of trials L, sampling budget per trial T, base failure rate δ_0 for $\ell = 1, 2..., L$ do Run algorithm \mathcal{A} until it self-terminates or exhausts the sampling budget T. if \mathcal{A} has self-terminated then $\hat{J}_{\ell} = J(\mathcal{A})$ else $\hat{J}_{\ell} = 0$ end if end for $\{0, 1, \ldots, K\}, v_i$ Count the votes: $\forall i \in$ $\sum_{\ell=1}^{L} \mathbb{1}\{\hat{J}_{\ell}=i\}.$ if $v_0 \ge \lfloor \frac{L}{2} \rfloor + 1$ then return 0 // failure else

// success

represents the stage (r, c), and the label below each dot represents the number of trials $(L_{r,c})$ and the budget assigned to each trials $(T_{r,c})$. For a fixed row r, each stage spends the same sampling budget of $L_{r,c}T_{r,c} =$ $r2^{r-1}L_1T_1$. However, the assignment of $L_{r,c}$ halves with increasing c and $T_{r,c}$ doubles with increasing c. For a fixed column c, as the row index increases, the total budget for each stage $L_{r,c}T_{r,c}$ doubles – in fact a slightly more than double due to a technical reason that will become clear in the proof of Theorem 11. Both the number of trials and budget increases row wise, meanwhile different combination of trial size and budget are employed column wise, keeping the effective total samples used constant throughout the a same row.

Our 2D doubling trick extends the vanilla doubling trick introduced in DSH. The vanilla doubling trick works by iterating through an input parameter (e.g., budget in DSH) at an exponentially growing rate to eventually reach the optimal value with only a logarithmic cost. Similarly, our 2D doubling trick in Algorithm 2 requires two input parameters, necessitating the design of a doubling scheme for both. While the budget doubling mirrors DSH, the voting scheme enhances confidence through the independence of repeated trials. Additionally, our choice of L_1 ensures the minimal number of repeated trials needed to achieve a confidence certificate of δ , as presented in the following theorem.

Theoretical analysis. We first show that Brake-Booster is δ -correct.

Theorem 11 (Correctness). Let an algorithm \mathcal{A} be δ -correct and have a sample complexity of $T^*_{\delta}(\mathcal{A})$. Suppose we run BrakeBooster (Algorithm 2) denoted by \mathcal{M} with input \mathcal{A} , $L_1 = \lceil \frac{4 \log(1+\frac{2}{\delta})}{\log \frac{1}{4e\delta_0}} \rceil$, $T_1 \ge 1$, and $\delta_0 \le \frac{1}{(2e)^2}$. Then,

$$\mathbb{P}\left(\tau(\mathcal{M}) < \infty, J(\mathcal{M}) \neq 1\right) \leq \delta.$$

The proof can be found at the end of this section. Furthermore, BrakeBooster enjoys an exponential stopping tail.

Theorem 12 (Exponential tail bound). Let an algorithm \mathcal{A} be δ -correct and have a high probability sample complexity of $T^*_{\delta}(\mathcal{A})$. Suppose we run Brake-Booster (Algorithm 2) with input \mathcal{A} , $L_1 = \lceil 4 \log(1 + \frac{2}{\delta}) \rceil$, $T_1 \geq 1$, and $\delta_0 = (\frac{1}{2e})^2$. Then, there exists $T_0 = \tilde{\Theta}((T^*_{\delta_0}(\mathcal{A}) + T_1) \cdot \ln(1/\delta))$ such that

$$\forall T \ge T_0, \ \mathbb{P}\left(\tau(\mathcal{M}) \ge T\right) \le \exp\left(-\frac{T}{T^*_{\delta_0}(\mathcal{A}) \cdot O(\log T)}\right)$$

That is, if T_1 is an absolute constant, then BrakeBooster enjoys a $(\tilde{\Theta}(T^*_{\delta_0}(\mathcal{A})\ln(1/\delta)), T^*_{\delta_0}(\mathcal{A}))$ -exponential stopping tail.

The theorem above literally delivers the promised guarantee – it takes in an algorithm with a high probability sample complexity guarantee $T^*_{\delta}(\mathcal{A})$ and turns in into the one that enjoys an exponential stopping tail without loosing the same high probability sample complexity guarantee due to Proposition 8(i), up to polylog(T) factors. For example, one can use Successive Elimination (Even-Dar et al., 2006) to obtain the following guarantee.

Corollary 13. Suppose we take Successive Elimination algorithm (Even-Dar et al., 2006) as \mathcal{A} and run BrakeBooster algorithm with $L_1 = \left\lceil \frac{4 \log(1+\frac{2}{\delta})}{\log \frac{1}{4e\delta_0}} \right\rceil$, $T_1 \geq 1$, and $\delta_0 = \left(\frac{1}{2e}\right)^2$. Then, BrakeBooster enjoys a $\left(\tilde{\Theta}\left(H_1 \ln(1/\delta)\right), \tilde{\mathcal{O}}(H_1)\right)$ -exponential stopping tail.

Proof of Theorem 11. In this proof, acute readers will notice that we often talk about events that happen in stage (r, c) without having a condition that the algorithm has not stopped before. To deal with this without notational overload, we take the model where the algorithm has already been run for all stages with-

out stopping, and the user of the algorithm only reveals what happened already and stop when the stopping condition is met. This way, we can talk about events in any stage without adding conditions on whether the algorithm has stopped or not (and this is valid because the samples are independent between stages).

Define

 $Q_{r,c} := \{\ell \in [L_{r,c}] : \ell\text{-th trial} \\ \text{self-terminates and output incorrect arm} \}.$

Note that

$$\mathbb{P}\left(\tau(\mathcal{M}) < \infty, J(\mathcal{M}) \neq 1\right)$$

= $\sum_{r=1}^{\infty} \sum_{c=1}^{r} \mathbb{P}\left(\mathcal{E}_{1}, \mathcal{E}_{2}\right) \cdot \mathbb{P}\left(J_{r,c} \notin \{0, 1\}\right)$
 $\leq \sum_{r=1}^{\infty} \sum_{c=1}^{r} \mathbb{P}\left(J_{r,c} \notin \{0, 1\}\right),$

where, $\mathcal{E}_1, \mathcal{E}_2$ are shorthand for

$$\mathcal{E}_1 := \{ \forall u \in [r-1], \ v \in [u], \ \hat{J}_{u,v} = 0 \},\$$

$$\mathcal{E}_2 := \{ \forall w \in [c-1], \hat{J}_{r,w} = 0 \}.$$

A stage outputting an incorrect arm index means that

- 1. More than half of the trials of \mathcal{A} has self-terminated; i.e., $v_0 \leq \lfloor L_{r,c}/2 \rfloor$ and thus $\sum_{i \in [K]} v_i \geq L_{r,c} \lfloor L_{r,c}/2 \rfloor$.
- 2. The majority vote of those terminated trials is an incorrect arm index; i.e., $\sum_{i \in [2..K]} v_i \geq \lfloor \frac{1}{2} \sum_{i \in [K]} v_i \rfloor$.

Thus, we have

$$|Q_{r,c}| = \sum_{i \in [2..K]} v_i \ge \left\lceil \frac{1}{2} (L_{r,c} - \lfloor L_{r,c}/2 \rfloor) \right\rceil \ge \left\lceil \frac{L_{r,c}}{4} \right\rceil,$$

which implies that

$$\sum_{r=1}^{\infty} \sum_{c=1}^{r} \mathbb{P}\left(J_{r,c} \notin \{0,1\}\right) \le \sum_{r=1}^{\infty} \sum_{c=1}^{r} \mathbb{P}\left(\left|Q_{r,c}\right| \ge \left\lceil \frac{L_{r,c}}{4} \right\rceil\right)$$

Assumption 1 implies that

 $\mathbb{P}(\text{a trial self-terminates and outputs an incorrect arm}) < \delta_0.$

Hence by Lemma 28 in the appendix with $\delta = \delta_0$ whose requirement $\delta_0 \leq \alpha = \frac{1}{4}$ is satisfied by the assumption of the theorem,

$$\begin{split} &\sum_{r=1}^{\infty} \sum_{c=1}^{r} \mathbb{P}\left(|Q_{r,c}| \ge \left\lceil \frac{L_{r,c}}{4} \right\rceil \right) \\ &\le \sum_{r=1}^{\infty} \sum_{c=1}^{r} \exp\left(-\frac{L_{r,c}}{4} \log(\frac{1}{4e\delta_0}) \right) \\ &= \sum_{r=1}^{\infty} \sum_{c=1}^{r} \exp\left(-\frac{r \cdot 2^{(r-c)}L_1}{4} \log(\frac{1}{4e\delta_0}) \right) \end{split}$$

$$= \sum_{r=1}^{\infty} \sum_{k=0}^{r-1} \exp\left(-\frac{r \cdot 2^k L_1}{4} \log(\frac{1}{4e\delta_0})\right)$$

Since $L_1 = \lceil \frac{4 \log(1 + \frac{4}{\delta})}{\log \frac{1}{4c\delta_0}} \rceil$, by evaluating an infinite sum (see Lemma 27), we conclude the proof:

$$\sum_{r=1}^{\infty} \sum_{k=0}^{r-1} \exp\left(-\frac{r \cdot 2^k L_1}{4} \log(\frac{1}{4e\delta_0})\right)$$

$$\leq \sum_{r=1}^{\infty} \frac{3}{2} \exp\left(-r \log(1+\frac{2}{\delta})\right)$$

$$= \frac{3}{2} \cdot \frac{\exp\left(-\log(1+\frac{2}{\delta})\right)}{1-\exp\left(-\log(1+\frac{2}{\delta})\right)} \qquad (\text{geometric sum})$$

$$\leq \frac{3}{2} \cdot \frac{\delta}{2} \leq \delta.$$

One can see from above the reason why we set $L_{r,c}$ to be $\propto r2^{r-1}$ rather than $\propto 2^{r-1}$ in the algorithm – without the extra factor of r, the sum of will not be controlled.

5 Related Work

While research on best arm identification can be traced back to the seminal work of Chernoff (1959), algorithms with fixed confidence with the correctness and the sample complexity guarantees first appeared in Even-Dar et al. (2006) where the authors proposed two influential algorithms: Median Elimination and Successive Elimination. The former shows a deterministic sample complexity of $\mathcal{O}\left(\frac{K}{\varepsilon^2}\log\frac{1}{\delta}\right)$ for finding an arm that is ε close to the best arm with a probability of at least $1-\delta$. This worst-case result is optimal up to a constant factor. The second algorithm, Successive Elimination, shows that, with a probability of at least $1 - \delta$, the sample complexity of identifying the best arm scales with an instance-dependent quantity $H_1 = \sum_{i=2}^{K} \frac{1}{\Delta_i^2}$, where Δ_i is the gap between the best arm and the *i*-th best arm. This result has had a significant influence on subsequent research on best arm identification. Since then, many algorithms have been proposed to improve the sample complexity of best arm identification in the fixed confidence setting. For instance, Kalyanakrishnan et al. (2012) propose the LUCB algorithm that extends Even-Dar et al. (2006) to the scenario where the algorithm is required to return the best m arms instead of just the best arm. Karnin et al. (2013b) and Jamieson et al. (2014) then propose algorithms with improved guarantees that turns problem-dependent logarithmic factors into doubly-logarithmic ones. Chen et al. (2017) have further improved both lower and upper bound guarantees on the high probability sample complexity. Garivier and Kaufmann (2016) propose the Track-and-Stop algorithm that asymptotically matches the asymptotic lower bound for its sample complex-



Figure 2: A diagram showing the progression of the stages in Algorithm 2 where each stage (r, c) has a different trial count $L_{r,c} = r2^{r-c}L_1$ and a per-trial budget $T_{r,c} = 2^{c-1}T_1$.

ity. However, notably, all these results focus on the expected or high-probability sample complexity, rather than the stopping time distribution. Therefore, these algorithms do not (yet) have a guarantee showing a light tail for the stopping time distribution, which does not exclude the possibility of running for a long time before stopping with a non-trivial probability.

The fixed budget algorithm, in contrast, must return the best arm within a pre-specified budget. Audibert et al. (2010) propose the first fixed budget algorithm called Successive Rejects. They show that the probability of misidentifying the best arm scales with a problem-dependent quantity, $H_2 = \max_{i=2}^{K} \frac{i}{\Delta^2}$. Karnin et al. (2013b) later improve this result by a logarithmic factor with an improved algorithm called Sequential Halving, which has been widely adopted in many applications including hyperparameter optimization Li et al. (2018). A recent work by Zhao et al. (2023) build upon Sequential Halving, addressing how to measure if the algorithm's output is good enough for any suboptimality gap ε , while ε is free to be chosen after the algorithm finishes. Additionally, this work extends Sequential Halving to the challenging data-poor regime, where the number of samples is even smaller than the number of arms. While we leverage some fixed budget algorithms such as Sequential Halving, the main focus of this paper is the fixed confidence setting.

6 Conclusion

We have provided a new perspective on the behavior of the stopping time for the fixed confidence best arm identification algorithms, which inspires numerous open problems. First, both of our proposed algorithms introduce nontrivial extra constant or logarithmic factors in their sample complexity compared to the wellknown optimal instance-dependent sample complexity that is achieved by existing algorithms (Garivier and Kaufmann, 2016). It would be interesting to investigate whether it is possible to develop novel algorithms that obtain the optimal instance-dependent optimality while attaining exponentially decaying tail bounds for the stopping time distribution. Second, the resetting mechanism employed by our algorithms tends to be less practical. It would be interesting to investigate whether there exists a simple and/or elegant algorithm that can avoid the resetting mechanism and exhibit practical numerical performance. In particular, studying whether or not the recently proposed practical algorithms in Jourdan et al. (2022) achieve exponential tail bounds and, if not, developing remedies for them would be an interesting avenue of research. Finally, it would be interesting to attain similar exponential tail bounds for more complex settings such as combinatorial bandits or Markov decision processes.

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Supplementary Materials

Appendix

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the questionnaire for submission: The previous reviewers recognized that our problem is important and has been surprisingly overlooked by the bandit community. However, one reviewer misunderstood part of the proof and thought there was an error due to some unclear writing. We've made significant improvements to the clarity of the paper for this submission.

A Lower bounds

```
\begin{array}{l} \label{eq:algorithm} \begin{array}{l} \textbf{Algorithm 4 Successive Elimination (Even-Dar et al., 2006)} \\ \hline \textbf{Input: } \delta \\ \hline \textbf{Initialize } t = 1, \ \mathcal{S} = [K], \ \hat{\mu}_i = 0 \ \text{for } \forall i \in [K] \\ \hline \textbf{Sample each arm } i \in \mathcal{S} \ \text{once} \\ \textbf{while } |\mathcal{S}| > 1 \ \textbf{do} \\ \hline \textbf{Sample each arm } i \in \mathcal{S} \ \text{once and update } \hat{\mu}_i \\ \hline \textbf{Set } S = S \setminus \left\{ i : \max_{j \in \mathcal{S}} \hat{\mu}_j - \hat{\mu}_i \geq \sqrt{\frac{2 \ln(3.3t^2/\delta)}{t}} \right\} \\ t = t+1 \\ \textbf{end while} \end{array}
```

Theorem 4 (Lower bound). For Successive Elimination, there exists an instance with a unique best arm where the algorithm never stops with a constant probability.

Proof. Consider an instance with K = 3 arms following Gaussian distributions $\mathcal{N}(1,1)$, $\mathcal{N}(0.9,1)$, and $\mathcal{N}(0.9,1)$. The lower bound we show here is a consequence of the potential misbehave of the algorithm after it pulls each arm for once. Since all of the arms have same variance, we simply denote the any confidence width, with a probability δ , of the sample mean after being pulled for t times as $b_{\delta,t} = \sqrt{\frac{2\ln(3.3t^2/\delta)}{t}}$ by Lemma 15. We define the following events:

• $E_1 := \{\hat{\mu}_{1,1} \le \mu_2 - 3b_{\delta,1}\}.$

•
$$E_2 := \left\{ \forall t > 0, \hat{\mu}_{2,t} \in \left(\mu_2 - b_{\delta,t}, \mu_2 + b_{\delta,t} \right) \right\}.$$

•
$$E_3 := \left\{ \forall t > 0, \hat{\mu}_{3,t} \in \left(\mu_3 - b_{\delta,t}, \mu_3 + b_{\delta,t} \right) \right\}.$$

We first note that the event $E_1 \cap E_2 \cap E_3$ implies that the algorithm eliminates the best arm after it pulls each arm for once and never stops. To see this, we have

$$\hat{\mu}_{1,1} \le \mu_2 - 3b_{\delta,1} = \mu_2 - b_{\delta,1} - 2b_{\delta,1} \le \hat{\mu}_{2,1} - 2b_{\delta,1}.$$

This implies that

$$\hat{\mu}_{1,1} + b_{\delta,1} \le \hat{\mu}_{2,1} - b_{\delta,1}$$

which is the condition for the algorithm to eliminate the best arm. Next we lower bound the probabilities of the events E_1 , E_2 , and E_3 . For the event E_1 , by Lemma 14,

$$\mathbb{P}(E_{1}) = \mathbb{P}\left(\hat{\mu}_{1,1} \le \mu_{2} - 3b_{\delta,1}\right) \\
= \mathbb{P}\left(\mu_{1} - \hat{\mu}_{1,1} > \Delta_{2} + 3b_{\delta,1}\right) \\
> \frac{1}{8\sqrt{\pi}} \exp\left(-\frac{7\left(\Delta_{2} + 3b_{\delta,1}\right)^{2}}{2}\right) \\
> \frac{1}{8\sqrt{\pi}} \exp\left(-\frac{7\left(4b_{\delta,1}\right)^{2}}{2}\right) \\
> \frac{1}{8\sqrt{\pi}} \left(\frac{\delta}{3.3}\right)^{118}.$$

By Lemma 15, we have for $\delta \leq 1/2$,

$$\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(E_2) \mathbb{P}(E_3) \ge (1-\delta)^2 \ge \frac{1}{4}$$

Thus, we have

$$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1) \mathbb{P}(E_2) \mathbb{P}(E_3) > \frac{1}{32\sqrt{\pi}} \left(\frac{\delta}{3.3}\right)^{118}$$

Thus with a constant probability, the algorithm never stops.

Lemma 14 (Anti-concentration inequality Abramowitz and Stegun (1968)). For a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ and any z > 0, we have

$$\mathbb{P}\left(|X-\mu| > z\sigma\right) > \frac{1}{4\sqrt{\pi}} \exp\left(-\frac{7z^2}{2}\right).$$

Lemma 15 (Naive anytime confidence bound). For a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, the sample mean $\hat{\mu}_t$ satisfies

$$\mathbb{P}\left(\forall t > 0, \hat{\mu}_t \in \left(\mu - \sqrt{\frac{2\sigma^2 \ln\left(3.3t^2/\delta\right)}{t}}, \mu + \sqrt{\frac{2\sigma^2 \ln\left(3.3t^2/\delta\right)}{t}}\right)\right) \ge 1 - \delta.$$

Algorithm 5 KL-LUCB (Tanczos et al., 2017), adapted and simplified for sub-Gaussian distribution

Input: δ Define: $L_i(T_i(t), \delta) = \hat{\mu}_i - b_{\delta, T_i(t)}$ and $U_i(T_i(t), \delta) = \hat{\mu}_i + b_{\delta, T_i(t)}$ Initialize: Sample each arm once while $L_{i^*}(T_{i^*}(t), \delta) < \max_{i \neq i^*} U_i(T_i(t), \delta)$ do Sample the following two arms: • $i^* = \arg \max_{i \in [K]} \hat{\mu}_i$

• $i' = \arg \max_{i \neq i^*} U_i(T_i(t), \delta)$

Update sample mean and confidence interval $t \leftarrow t+2$ end while

Theorem 5. For KL-LUCB (Algorithm 5), there exists an instance with a unique best arm where the algorithm never stops with a constant probability.

Proof. The proof follows the same line as the proof of Theorem 4. We consider an instance with K = 3 arms following Gaussian distributions $\mathcal{N}(1,1)$, $\mathcal{N}(0.9,1)$, and $\mathcal{N}(0.9,1)$. We define the following events:

• $E_1 := \{\hat{\mu}_{1,1} \le \mu_2 - 3b_{\delta,1}\}.$

•
$$E_2 := \left\{ \forall t > 0, \hat{\mu}_{2,t} \in \left(\mu_2 - b_{\delta,t}, \mu_2 + b_{\delta,t} \right) \right\}.$$

• $E_3 := \left\{ \forall t > 0, \hat{\mu}_{3,t} \in \left(\mu_3 - b_{\delta,t}, \mu_3 + b_{\delta,t} \right) \right\}.$

We first note that the event $E_1 \cap E_2 \cap E_3$ implies that the algorithm will never pull the best arm after it pulls each arm for once and never stops. To see this, we first show that the best arm has lowest sample mean among the three arms given $E_1 \cap E_2 \cap E_3$. We take the second arm as an example, but same argument for the third arm,

$$\hat{\mu}_{1,1} \le \mu_2 - 3b_{\delta,1} = \mu_2 - b_{\delta,1} - 2b_{\delta,1} \le \hat{\mu}_{2,1} - 2b_{\delta,1} < \hat{\mu}_{2,1}$$

Thereby the algorithm will take arg $\max_{i=2,3} \hat{\mu}_i$ as the best arm and pull both of arm 2 and 3 in the next round. Given the event $E_2 \cap E_3$, the confidence interval for arm 2 and 3 keep shrinking, while the confidence interval for arm 1 keeps still. Thus arm 1 will never be pulled again. Also given $E_2 \cap E_3$, the confidence interval for arm 2 and 3 will always overlap, thus the algorithm will never stop.

The lower bound of the probabilities of the events $E_1 \cap E_2 \cap E_3$ is same as the proof of Theorem 4.

B Doubling Sequential Halving with fixed confidence (FC-DSH)

Throughout this section, acute readers will notice that we may talk about events that happen in phase m without having a condition that the algorithm has not stopped before. To deal with this without notational overload, we take the model where the algorithm has already been run for all phases without stopping, and the user of the algorithm only reveals what happened already and stop when the stopping condition is met. This way, we can talk about events in any phase without adding conditions on whether the algorithm has stopped or not (and this is valid because the samples are independent between phases).

B.1 Proof of Theorem of FC-DSH's Correctness

Theorem 9 (Correctness). FC-DSH runs with confidence level δ is δ -correct.

Proof. Following the definition of δ -correction in 1, we need to prove that $\mathbb{P}(\tau(\text{DSH}) < \infty, J(\text{DSH}) \neq 1) \leq \delta.$

For each phase $m \in \{1, 2, ...\}$ in FC-DSH, for each stage $\ell \in \left[\left\lceil \log_2(K) \right\rceil \right]$, for each arm $i \in [K]$, recall the confidence width

$$b_i^{(m)} = \sqrt{\frac{2}{N^{(m,\ell_i)}} \log\left(\frac{6K \left\lceil \log_2(K) \right\rceil m^2}{\delta}\right)}$$

and define the following events:

 $\mathbb{P}(G^c)$

and

$$G := \left\{ \forall i \in [K], \forall m \in \{1, 2, \ldots\}, \forall \ell \in \left[\left\lceil \log_2(K) \right\rceil \right], \left| \hat{\mu}_i^{(m,\ell)} - \mu_i \right| \le b_i^{(m,\ell)} \right\}, \\ G^c := \left\{ \exists i \in [K], \exists m \in \{1, 2, \ldots\}, \exists \ell \in \left[\left\lceil \log_2(K) \right\rceil \right], \left| \hat{\mu}_i^{(m,\ell)} - \mu_i \right| > b_i^{(m,\ell)} \right\}.$$

We first claim that the probability of event G^c happens is at most δ . The proof is as follows

$$\leq \sum_{m=1}^{\infty} \sum_{\ell=1}^{\lceil \log_2(K) \rceil} \sum_{i=1}^{K} \mathbb{P}\left(\left| \hat{\mu}_i^{(m,\ell)} - \mu_i \right| > \sqrt{\frac{2}{N_i^{(m,\ell)}} \log\left(\frac{6K \left\lceil \log_2(K) \right\rceil m^2}{\delta}\right)} \right)$$
(use union bound)
$$\leq \sum_{m=1}^{\infty} \sum_{\ell=1}^{\lceil \log_2(K) \rceil} \sum_{i=1}^{K} 2 \exp\left(-\frac{N_i^{(m,\ell)} \left(\sqrt{\frac{2}{N_i^{(m,\ell)}} \log\left(\frac{6K \left\lceil \log_2(K) \right\rceil m^2}{\delta}\right)} \right)^2}{2} \right)$$
(use Hoeffding's inequality)
$$= \sum_{m=1}^{\infty} \sum_{\ell=1}^{\lceil \log_2(K) \rceil} \sum_{i=1}^{K} \frac{2\delta}{6K \left\lceil \log_2(K) \right\rceil m^2}$$
$$= \sum_{m=1}^{\infty} \sum_{\ell=1}^{\lceil \log_2(K) \rceil} \sum_{i=1}^{K} \frac{\delta}{3K \left\lceil \log_2(K) \right\rceil m^2}$$
$$= \sum_{m=1}^{\infty} \frac{\delta}{3m^2}$$
(use geometric sum $\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$)

 $\leq \delta$.

Secondly, we claim that under the event G, FC-DSH never outputs a suboptimal arm, formally, $\mathbb{P}(J \neq 1, G) = 0$. We prove this claim by contradiction.

Suppose FC-DSH outputs arm $J \neq 1$. From the stopping condition $L_J^{(m)} \geq \max_i U_{i \neq J}^{(m)}$,

$$L_J^{(m)} \ge \max_i U_{i \neq J}^{(m)}$$

$$\Rightarrow L_J^{(m)} \ge U_1^{(m)}$$

$$\Rightarrow \hat{\mu}_J^{(m)} - b_J^{(m)} \ge \hat{\mu}_1^{(m)} + b_1^{(m)}.$$

Under event G, we have $\mu_J \ge \hat{\mu}_J^{(m)} - b_J^{(m)}$ and $\hat{\mu}_1^{(m)} + b_1^{(m)} \ge \mu_1$, which implies $\mu_J \ge \hat{\mu}_J^{(m)} - b_J^{(m)} \ge \hat{\mu}_1^{(m)} + b_1^{(m)} \ge \mu_1$.

Therefore, we have $\mu_J \ge \mu_1$ which is a contradiction.

Finally, we combine both claims to show

$$\mathbb{P}(\tau < \infty, J \neq 1)$$

= $\mathbb{P}(\tau < \infty, J \neq 1, G^c) + \mathbb{P}(\tau < \infty, J \neq 1, G)$
 $\leq \mathbb{P}(G^c) + \mathbb{P}(J \neq 1, G)$
 $\leq \delta + 0 = \delta.$

B.2 Proof of Theorem of FC-DSH's Tail Bound

Theorem 10 (Exponential tail bound). Let $T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$ and $\kappa = 4096H_2 \log_2(K)$. Then, for all $T \ge T_{\delta}$, FC-DSH satisfies

$$\mathbb{P}\left(\tau \geq T\right) \leq \exp\left(-\frac{T}{2\kappa}\right).$$

Proof. To avoid redundancy and for the sake of readability, from now on, we assume K is of a power of 2. Hence $\lceil \log_2(K) \rceil = \log_2(K)$. It is easy to verify the result for any K.

Denote by $m_{\tau} = \min\left\{m \ge 1: L_{J_m}^{(m)} \ge \max_{i \ne J_m} U_i^{(m)}\right\}$ the stopping phase of FC-DSH. Then, τ corresponds to the total number of samples at the end of stopping phase m_{τ} .

It suffices to show for all phase m such that $T_m \ge T_{\delta}$, $\mathbb{P}(\tau \ge T_m) \le \exp\left(-\frac{T_m}{\kappa}\right)$. Then, we can apply Lemma 16 to obtain for all $T \ge T_{\delta}$, $\mathbb{P}(\tau \ge T) \le \exp\left(-\frac{T}{2\kappa}\right)$.

Recall the stopping condition: at the end of phase m, FC-DSH outputs an arm J_m , and FC-DSH will stop if the condition $L_{J_m}^{(m)} \ge \max_{i \ne J_m} U_i^{(m)}$ is satisfied.

Let m be any phase such that $T_m \ge T_{\delta}$. We start off by decomposing probability bound according to the stopping condition

$$\mathbb{P}(\tau \ge T_m)$$

$$\le \prod_{m'=1}^m \mathbb{P}(\text{phase } m' \text{ failed to stop})$$

$$\le \mathbb{P}(\text{phase } m \text{ failed to stop})$$

$$= \mathbb{P}\left(L_{J_m}^{(m)} < \max_{i \ne J_m} U_i^{(m)}\right)$$

$$= \mathbb{P}\left(L_{J_m}^{(m)} < \max_{i \neq J_m} U_i^{(m)}, J_m \neq 1\right) + \mathbb{P}\left(L_{J_m}^{(m)} < \max_{i \neq J_m} U_i^{(m)}, J_m = 1\right)$$

$$\leq \mathbb{P}\left(L_{J_m}^{(m)} < \max_{i \neq J_m} U_i^{(m)}, J_m \neq 1\right) + \sum_{i \neq 1} \mathbb{P}\left(L_{J_m}^{(m)} < U_i^{(m)}, J_m = 1\right).$$

For the first term, we obtain the following probability bound

$$\mathbb{P}\left(L_{J_m}^{(m)} < \max_{i \neq J_m} U_i^{(m)}, J_m \neq 1\right)$$

$$\leq \mathbb{P}\left(J_m \neq 1\right)$$

$$\leq \exp\left(-\frac{T_m}{512H_2\log_2(K)}\right). \qquad (\text{use Lemma 20})$$

In phase m, for each $i \in [K]$, we denote by ℓ_i the stage at which arm i is eliminated. We define ℓ_i^* to be the largest stage ℓ_i such that $\Delta_i \leq \frac{1}{2}\Delta_{\frac{K}{4},2^{-\ell_i+1}}$. We observe that there are arms i that have no such ℓ_i^* and there are arms i that ℓ_i^* exists. For the former case, if an arm i has no such ℓ_i^* , it implies $\Delta_i > \Delta_{\frac{K}{4}}$. We further split the second term accordingly

$$\begin{split} &\sum_{i \neq 1} \mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \ J_m = 1 \right) \\ &\leq \sum_{\substack{i \neq 1: \\ \Delta_i > \Delta_{K_4}}} \mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \ J_m = 1 \right) + \sum_{\substack{i \neq 1: \\ \ell_i^* \text{ exists}}} \mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \ J_m = 1 \right) \\ &\leq \sum_{\substack{i \neq 1: \\ \Delta_i > \Delta_{K_4}}} \mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \ J_m = 1 \right) \\ &+ \sum_{\substack{i \neq 1: \\ \ell_i^* \text{ exists}}} \left(\mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \ J_m = 1, \ \ell_i \ge \ell_i^* \right) + \mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \ J_m = 1, \ \ell_i < \ell_i^* \right) \right). \end{split}$$

For the first subterm, for all arm *i* that satisfies $i \neq 1$, $\Delta_i > \frac{1}{2}\Delta_{\frac{K}{4}}$, we apply Lemma Lemma 17 to obtain the following probability bound

$$\mathbb{P}\left(L_1^{(m)} < U_i^{(m)}, J_m = 1\right) \le \exp\left(-\frac{T_m}{512H_2\log_2(K)}\right)$$

For the second and third subterms, for all arm i that satisfies $i \neq 1$ and ℓ_i^* exists, we apply Lemma 18 to obtain the following probability bound

$$\mathbb{P}\left(L_1^{(m)} < U_i^{(m)}, \, \ell_i \ge \ell_i^*, \, J_m = 1\right) \le \exp\left(-\frac{T_m}{1024H_2\log_2(K)}\right),$$

and we apply Lemma 19 to obtain the following probability bound

$$\mathbb{P}\left(\ell_i < \ell_i^*, J_m = 1\right) \le \exp\left(-\frac{T_m}{2048H_2\log_2(K)}\right)$$

Hence, for the second term, we obtain the following probability bound

$$\sum_{i \neq 1} \left(\mathbb{P} \left(L_1^{(m)} < U_i^{(m)}, \, \ell_i \ge \ell_i^*, \, J_m = 1 \right) + \mathbb{P} \left(\ell_i < \ell_i^*, \, J_m = 1 \right) \right)$$

$$\leq \sum_{\substack{i \neq 1: \\ \Delta_i > \Delta_K^-}} \mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \, J_m = 1 \right)$$

$$+ \sum_{\substack{i \neq 1: \\ \ell_i^* \text{ exists}}} \left(\mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \, J_m = 1, \, \ell_i \ge \ell_i^* \right) + \mathbb{P} \left(L_{J_m}^{(m)} < U_i^{(m)}, \, J_m = 1, \, \ell_i < \ell_i^* \right) \right)$$

$$\leq \sum_{\substack{i \neq 1: \\ \Delta_i > \Delta_{\frac{K}{4}}}} \exp\left(-\frac{T_m}{512H_2 \log_2(K)}\right) \\ + \sum_{\substack{i \neq 1: \\ \ell_i^* \text{ exists}}} \left(\exp\left(-\frac{T_m}{1024H_2 \log_2(K)}\right) + \exp\left(-\frac{T_m}{2048H_2 \log_2(K)}\right)\right) \\ \leq 2\sum_{i \neq 1} \exp\left(-\frac{T_m}{2048H_2 \log_2(K)}\right) \\ \leq 2(K-1) \exp\left(-\frac{T_m}{2048H_2 \log_2(K)}\right).$$

Finally, we combine two terms and obtain the probability bound

$$\mathbb{P}\left(L_{J_m}^{(m)} < \max_{i \neq J_m} U_i^{(m)}, J_m \neq 1\right) + \sum_{i \neq 1} \mathbb{P}\left(L_{J_m}^{(m)} < U_i^{(m)}, J_m = 1\right)$$

= $\exp\left(-\frac{T_m}{512H_2\log_2(K)}\right) + 2(K-1)\exp\left(-\frac{T_m}{2048H_2\log_2(K)}\right)$
 $\leq 2K \exp\left(-\frac{T_m}{2048H_2\log_2(K)}\right)$
= $\exp\left(-\frac{T_m}{2048H_2\log_2(K)} + \log(2K)\right).$

The condition $T_m \ge T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$ implies that

$$T_{m} \geq 4096H_{2}\log_{2}(K)\log\left(\frac{6K\log_{2}(K)}{\delta}\right)$$

$$\Rightarrow T_{m} \geq 4096H_{2}\log_{2}(K)\log(2K) \qquad (\text{use } \delta \leq 1)$$

$$\Rightarrow \log(2K) \leq \frac{T_{m}}{4096H_{2}\log_{2}(K)}$$

$$\Rightarrow -\frac{T_{m}}{2048H_{2}\log_{2}(K)} + \log(2K) \leq -\frac{T_{m}}{4096H_{2}\log_{2}(K)}.$$

Therefore,

$$\begin{split} & \exp\left(-\frac{T_m}{2048H_2\log_2(K)} + \log(2K)\right) \\ & \leq \exp\left(-\frac{T_m}{4096H_2\log_2(K)}\right), \end{split}$$

which concludes the proof of our theorem.

Lemma 16. Suppose for all phase m such that $T_m \ge T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$, for a constant c, FC-DSH achieves

$$\mathbb{P}\left(\tau \ge T_m\right) \le \exp\left(-\frac{T_m}{cH_2\log_2(K)}\right)$$

then, for all $T \geq T_{\delta}$, FC-DSH achieves

$$\mathbb{P}\left(\tau \ge T\right) \le \exp\left(-\frac{T}{2cH_2\log_2(K)}\right)$$

Proof. Let $T \geq T_{\delta}$. Then, there exists m such that

$$T_m \le T < T_{m+1}.$$

By the FC-DSH setup, $2T_m = T_{m+1}$, thus $T < T_{m+1} = 2T_m$

Then,

$$\mathbb{P}(\tau \ge T)$$

$$\leq \mathbb{P}(\tau \ge T_m) \qquad (\text{use } T \ge T_m)$$

$$\leq \exp\left(-\frac{T_m}{cH_2\log_2(K)}\right)$$

$$\leq \exp\left(-\frac{T}{2cH_2\log_2(K)}\right). \qquad (\text{use } T < 2T_m)$$

Lemma 17. Let u be any arm that satisfies $u \neq 1$, $\Delta_u > \frac{1}{2}\Delta_{\frac{K}{4}}$. For all phase m such that $T_m \geq T_{\delta} = 4096H_2\log_2(K)\log\left(\frac{6K\log_2(K)}{\delta}\right)$, we obtain

$$\mathbb{P}\left(L_1^{(m)} < U_u^{(m)}, J_m = 1\right) \le \exp\left(-\frac{T_m}{512H_2\log_2(K)}\right).$$

Proof. In the following proof, we abbreviate $\log(\cdot) = \log\left(\frac{6K\log_2(K)m^2}{\delta}\right)$ for clarity. We have

$$\mathbb{P}\left(L_{1}^{(m)} < U_{u}^{(m)}, J_{m} = 1\right) \\
= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \sqrt{\frac{2}{N^{(m,\ell_{1})}}\log(\cdot)} < \hat{\mu}_{u}^{(m)} + \sqrt{\frac{2}{N^{(m,\ell_{u})}}\log(\cdot)}, J_{m} = 1\right) \quad \text{(by the definition of } L_{1}^{(m)} \text{ and } U_{1}^{(m)} \text{)} \\
= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \hat{\mu}_{u}^{(m)} < \sqrt{\frac{2}{N^{(m,\ell_{u})}}\log(\cdot)} + \sqrt{\frac{2}{N^{(m,\ell_{1})}}\log(\cdot)}, J_{m} = 1\right) \\
\leq \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \hat{\mu}_{u}^{(m)} < 2\sqrt{\frac{2}{N^{(m,\ell_{u})}}\log(\cdot)}, J_{m} = 1\right) \quad \text{(since } J_{m} = 1, N^{(m,\ell_{1})} \ge N^{(m,\ell_{u})}) \\
= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\Delta_{u} + 2\sqrt{\frac{2}{N^{(m,\ell_{u})}}\log(\cdot)}, J_{m} = 1\right).$$

Lemma 21 states that for all arm u such that $u \neq 1$, $\Delta_u > \frac{1}{2}\Delta_{\frac{K}{4}}$, for all phase m such that $T_m \geq T_{\delta}$, $N^{(m,\ell_u)} \geq \frac{32}{\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$. This inequality implies that

$$\begin{split} N^{(m,\ell_u)} &\geq \frac{32}{\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right) \\ \Rightarrow \Delta_u^2 &\geq \frac{32}{N^{(m,\ell_u)}} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right) \\ \Rightarrow \Delta_u &\geq 4\sqrt{\frac{2}{N^{(m,\ell_u)}} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)} \\ \Rightarrow -\Delta_u + 2\sqrt{\frac{2}{N^{(m,\ell_u)}} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)} \leq -\frac{\Delta_u}{2}. \end{split}$$

Hence,

$$= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\Delta_{u} + 2\sqrt{\frac{2}{N^{(m,\ell_{u})}}\log\left(\frac{6K\log_{2}(K)m^{2}}{\delta}\right)}, J_{m} = 1\right)$$

$$\leq \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\frac{\Delta_{u}}{2}, J_{m} = 1\right)$$

$$\begin{split} &= \sum_{z=1}^{L} \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\frac{\Delta_{u}}{2}, \ell_{u} = z, J_{m} = 1\right) \\ &\leq \sum_{z=1}^{L} \exp\left(-\frac{\Delta_{u}^{2}}{2\left(\frac{1}{N^{(m,L)}} + \frac{1}{N^{(m,z)}}\right)}\right) \qquad (\text{use Hoeffding's inequality}) \\ &\leq \sum_{z=1}^{L} \exp\left(-\frac{\Delta_{u}^{2}N^{(m,1)}}{16}\right) \qquad (\text{use } L \ge 1 \text{ and } z \ge 1) \\ &= \sum_{z=1}^{L} \exp\left(-\frac{\Delta_{u}^{2}}{16} \cdot \frac{T_{m}}{K \log_{2}(K)}\right) \qquad (\text{by def } N^{(m,1)} = \frac{T_{m}}{K \log_{2}(K)}) \\ &\leq \sum_{z=1}^{L} \exp\left(-\frac{\Delta_{u}^{2}}{44} \cdot \frac{T_{m}}{K \log_{2}(K)}\right) \qquad (\text{use } \Delta_{u} > \frac{1}{2}\Delta_{\frac{K}{4}}\right) \\ &\leq \sum_{z=1}^{L} \exp\left(-\frac{\Delta_{u}^{2}}{\frac{K}{4}} \cdot \frac{T_{m}}{256 \log_{2}(K)}\right) \\ &\leq \sum_{z=1}^{L} \exp\left(-\frac{1}{H_{2}} \cdot \frac{T_{m}}{256 \log_{2}(K)}\right) \qquad (\text{by the definition } H_{2} = \max_{i} i\Delta_{i}^{-2}) \\ &= \sum_{z=1}^{L} \exp\left(-\frac{T_{m}}{256H_{2} \log_{2}(K)}\right) \\ &\leq \log_{2}(K) \exp\left(-\frac{T_{m}}{256H_{2} \log_{2}(K)}\right) \\ &\leq \exp\left(-\frac{T_{m}}{256H_{2} \log_{2}(K)} + \log(\log_{2}(K))\right). \end{aligned}$$

The condition
$$T_m \ge T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$$
 implies that
 $T_m \ge 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$
 $\Rightarrow T_m \ge 512H_2 \log_2(K) \log\left(\log_2(K)\right)$ (use $\delta \le 1$)
 $\Rightarrow \log\left(\log_2(K)\right) \le \frac{T_m}{512H_2 \log_2(K)}$
 $\Rightarrow -\frac{T_m}{256H_2 \log_2(K)} + \log\left(\log_2(K)\right) \le -\frac{T_m}{512H_2 \log_2(K)}.$
Therefore,

$$\exp\left(-\frac{T_m}{256H_2\log_2(K)} + \log(\log_2(K))\right)$$
$$\leq \exp\left(-\frac{T_m}{512H_2\log_2(K)}\right).$$

Lemma 18. Let $u \neq 1$ be any non-optimal arm. For all phase m such that $T_m \geq T_{\delta} = 4096H_2\log_2(K)\log\left(\frac{6K\log_2(K)}{\delta}\right)$, assuming ℓ_u^* exists, we obtain $\mathbb{P}\left(L_1^{(m)} < U_u^{(m)}, \, \ell_u \geq \ell_u^*, \, J_m = 1\right) \leq \exp\left(-\frac{T_m}{1024H_2\log_2(K)}\right).$

Proof. Recall that
$$\ell_u^*$$
 is defined to be the largest stage ℓ such that $\Delta_u \leq \frac{1}{2} \Delta_{\frac{K}{4} \cdot 2^{-\ell+1}}$. In the following proof, we

abbreviate $\log(\cdot) = \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$ for brevity.

We have

$$\mathbb{P}\left(L_{1}^{(m)} < U_{u}^{(m)}, \, \ell_{u} \ge \ell_{u}^{*}, \, J_{m} = 1\right) \\
= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \sqrt{\frac{2}{N^{(m,\ell_{1})}}\log(\cdot)} < \hat{\mu}_{u}^{(m)} + \sqrt{\frac{2}{N^{(m,\ell_{u})}}\log(\cdot)}, \, \ell_{u} \ge \ell_{u}^{*}, \, J_{m} = 1\right) \\$$
(by the definition of $L_{1}^{(m)}$ and $U_{1}^{(m)}$)

$$\leq \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \sqrt{\frac{2}{N^{(m,\ell_{1})}}\log(\cdot)} < \hat{\mu}_{u}^{(m)} + \sqrt{\frac{2}{N^{(m,\ell_{u}^{*})}}\log(\cdot)}, \ \ell_{u} \geq \ell_{u}^{*}, \ J_{m} = 1\right)$$
 (use $\ell_{u} \geq \ell_{u}^{*}$)
$$= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \hat{\mu}_{u}^{(m)} < \sqrt{\frac{2}{N^{(m,\ell_{u}^{*})}}\log(\cdot)} + \sqrt{\frac{2}{N^{(m,\ell_{u}^{*})}}\log(\cdot)}, \ \ell_{u} \geq \ell_{u}^{*}, \ J_{m} = 1\right)$$
(since $J_{m} = 1, \ N^{(m,\ell_{1})} \geq N^{(m,\ell_{u}^{*})}$)
$$= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\Delta_{u} + 2\sqrt{\frac{2}{N^{(m,\ell_{u}^{*})}}\log(\cdot)}, \ \ell_{u} \geq \ell_{u}^{*}, \ J_{m} = 1\right).$$

Lemma 22 states that for any arm u such that $u \neq 1$ and ℓ_u^* exists, for all phase m such that $T_m \geq T_{\delta}$, $N^{(m,\ell_u^*)} \geq \frac{32}{\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$. This inequality implies that

$$\begin{split} N^{(m,\ell_u^*)} &\geq \frac{32}{\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right) \\ \Rightarrow \Delta_u^2 &\geq \frac{32}{N^{(m,\ell_u^*)}} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right) \\ \Rightarrow \Delta_u &\geq 4\sqrt{\frac{2}{N^{(m,\ell_u^*)}} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)} \\ \Rightarrow -\Delta_u + 2\sqrt{\frac{2}{N^{(m,\ell_u^*)}} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)} \leq -\frac{\Delta_u}{2}. \end{split}$$

Hence,

$$\begin{split} &= \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\Delta_{u} + 2\sqrt{\frac{2}{N^{(m,\ell_{u}^{*})}}\log\left(\frac{6K\log_{2}(K)m^{2}}{\delta}\right)}, \ \ell_{u} \ge \ell_{u}^{*}, \ J_{m} = 1\right) \\ &\leq \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\frac{\Delta_{u}}{2}, \ \ell_{u} \ge \ell_{u}^{*}, \ J_{m} = 1\right) \\ &= \sum_{z=\ell_{u}^{*}}^{L} \mathbb{P}\left(\hat{\mu}_{1}^{(m)} - \mu_{1} - \hat{\mu}_{u}^{(m)} + \mu_{u} < -\frac{\Delta_{u}}{2}, \ \ell_{u} = z, \ J_{m} = 1\right) \\ &\leq \sum_{z=\ell_{u}^{*}}^{L} \exp\left(-\frac{\frac{\Delta_{u}^{2}}{4}}{2\left(\frac{1}{N^{(m,L)}} + \frac{1}{N^{(m,z)}}\right)}\right) \\ &\leq \sum_{z=\ell_{u}^{*}}^{L} \exp\left(-\frac{\Delta_{u}^{2}N^{(m,\ell_{u}^{*})}}{16}\right) \\ &= \sum_{z=\ell_{u}^{*}}^{L} \exp\left(-\frac{\Delta_{u}^{2}N^{(m,\ell_{u}^{*})}}{16}\right) \\ &= \sum_{z=\ell_{u}^{*}}^{L} \exp\left(-\frac{\Delta_{u}^{2}}{16} \cdot \frac{T_{m}}{2^{-\ell_{u}^{*}+1}K\log_{2}(K)}\right) \\ &\qquad (by \ def \ N^{(m,\ell_{u}^{*})} = \frac{T_{m}}{2^{-\ell_{u}^{*}+1}K\log_{2}(K)}) \end{split}$$

There are two cases: (1) $\frac{K}{4}2^{-\ell_u^*} \ge 1$ and (2) $\frac{K}{4}2^{-\ell_u^*} < 1$.

In the first case: by the definition of ℓ_u^* , we have $\frac{1}{2}\Delta_{\frac{K}{4}\cdot 2^{-\ell_u^*}} \leq \Delta_u$. Thus, we have

$$\exp\left(-\frac{\Delta_{u}^{2}}{16} \cdot \frac{T_{m}}{2^{-\ell_{u}^{*}+1} K \log_{2}(K)}\right)$$

$$\leq \exp\left(-\frac{\Delta_{\frac{K}{4}2^{-\ell_{u}^{*}}}^{2}}{64} \cdot \frac{T_{m}}{2^{-\ell_{u}^{*}+1} K \log_{2}(K)}\right)$$

$$= \exp\left(-\frac{\Delta_{\frac{K}{4}2^{-\ell_{u}^{*}}}^{2}}{\frac{K}{4}2^{-\ell_{u}^{*}}} \cdot \frac{T_{m}}{512 \log_{2}(K)}\right)$$

$$\leq \exp\left(-\frac{1}{H_{2}} \cdot \frac{T_{m}}{512 \log_{2}(K)}\right)$$

$$= \exp\left(-\frac{T_{m}}{512 H_{2} \log_{2}(K)}\right).$$

(by the definition $H_2 = \max_i i \Delta_i^{-2}$)

In the second case $\frac{K}{4}2^{-\ell_u^*} < 1$, we have

$$\exp\left(-\frac{\Delta_u^2}{16} \cdot \frac{T_m}{2^{-\ell_u^*+1} K \log_2(K)}\right)$$

$$= \exp\left(-\frac{\Delta_u^2}{128} \cdot \frac{T_m}{\frac{K}{4} 2^{-\ell_u^*} \log_2(K)}\right)$$

$$\leq \exp\left(-\frac{\Delta_u^2}{128} \cdot \frac{T_m}{\log_2(K)}\right) \qquad (\text{use } \frac{K}{4} 2^{-\ell_u^*} < 1)$$

$$\leq \exp\left(-\frac{\Delta_u^2}{u} \cdot \frac{T_m}{128 \log_2(K)}\right) \qquad (\text{use } u \ge 1)$$

$$\leq \exp\left(-\frac{1}{H_2} \cdot \frac{uT_m}{128 \log_2(K)}\right) \qquad (\text{by the definition } H_2 = \max_i i \Delta_i^{-2})$$

$$= \exp\left(-\frac{T_m}{128 H_2 \log_2(K)}\right)$$

$$\leq \exp\left(-\frac{T_m}{512 H_2 \log_2(K)}\right).$$

In both cases, we have

$$\exp\left(-\frac{\Delta_u^2}{16} \cdot \frac{T_m}{2^{-\ell_u^*+1} K \log_2(K)}\right)$$
$$\leq \exp\left(-\frac{T_m}{512H_2 \log_2(K)}\right).$$

Therefore,

$$\begin{split} &\sum_{z=\ell_u^*}^L \exp\left(-\frac{\Delta_u^2}{16} \cdot \frac{T_m}{2^{-\ell_u^*+1} K \log_2(K)}\right) \\ &\leq \sum_{z=\ell_u^*}^L \exp\left(-\frac{T_m}{512 H_2 \log_2(K)}\right) \\ &\leq \log_2(K) \exp\left(-\frac{T_m}{512 H_2 \log_2(K)}\right) \\ &\leq \exp\left(-\frac{T_m}{512 H_2 \log_2(K)} + \log(\log_2(K))\right). \end{split}$$

The condition
$$T_m \ge T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$$
 implies that

$$T_m \ge 4096H_2 \log_2(K) \log\left(\log_2(K)\right)$$

$$\Rightarrow T_m \ge 1024H_2 \log_2(K) \log\left(\log_2(K)\right)$$

$$\Rightarrow \log\left(\log_2(K)\right) \le \frac{T_m}{1024H_2 \log_2(K)}$$

$$\Rightarrow -\frac{T_m}{512H_2 \log_2(K)} + \log\left(\log_2(K)\right) \le -\frac{T_m}{1024H_2 \log_2(K)}.$$
Therefore,

$$\exp\left(-\frac{T_m}{512H_2 \log_2(K)} + \log(\log_2(K))\right)$$

$$\le \exp\left(-\frac{T_m}{1024H_2 \log_2(K)}\right).$$

Lemma 19. Let $u \neq 1$ be any arm non-optimal. For all phase m such that $T_m \geq T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$, assuming ℓ_u^* exists, we obtain

$$\mathbb{P}\left(\ell_u < \ell_u^*\right) \le \exp\left(-\frac{T_m}{2048H_2\log_2(K)}\right).$$

Proof. Recall that ℓ_u^* to be the largest stage ℓ such that $\Delta_u \leq \frac{1}{2} \Delta_{\frac{K}{4} \cdot 2^{-\ell+1}}$. We denote by \mathcal{A}_ℓ the set of arms at stage ℓ . In the event arm u does not survive until stage ℓ_u^* . We have

$$\begin{split} & \mathbb{P}\left(\ell_{u} < \ell_{u}^{*}\right) \\ &= \mathbb{P}\left(\exists \ell \in [\ell_{u}^{*} - 1] : u \notin \mathcal{A}_{\ell+1}, u \in \mathcal{A}_{\ell}\right) \\ &\leq \sum_{\ell=1}^{\ell_{u}^{*} - 1} \mathbb{P}\left(u \notin \mathcal{A}_{\ell+1}, u \in \mathcal{A}_{\ell}\right) \qquad (\text{use union bound}) \\ &\leq \sum_{\ell=1}^{\ell_{u}^{*} - 1} \sum_{\substack{a_{\ell} \subseteq [K]:\\ u \in a_{\ell},\\ |a_{\ell}| = K \cdot 2^{-\ell+1}}} \mathbb{P}\left(u \notin \mathcal{A}_{\ell+1}, u \in \mathcal{A}_{\ell}, \mathcal{A}_{\ell} = a_{\ell}\right) \qquad (\text{use the law of total probability}) \\ &= \sum_{\ell=1}^{\ell_{u}^{*} - 1} \sum_{\substack{a_{\ell} \subseteq [K]:\\ u \in a_{\ell},\\ |a_{\ell}| = K \cdot 2^{-\ell+1}}} \mathbb{P}\left(u \notin \mathcal{A}_{\ell+1}, u \in \mathcal{A}_{\ell} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right). \end{split}$$

By the setup of the FC-DSH, at stage ℓ , the event arm u is in set \mathcal{A}_{ℓ} but is eliminated next around means that there exists a set $A \subset \mathcal{A}_{\ell}$ that half the size of \mathcal{A}_{ℓ} such that $\forall i \in A$, $\hat{\mu}_u \leq \hat{\mu}_i$. Formally,

$$\sum_{\ell=1}^{\ell_u^*-1} \sum_{\substack{a_\ell \subseteq [K]:\\ u \in a_\ell, \\ |a_\ell| = K2^{-\ell+1}}} \mathbb{P}\left(u \notin \mathcal{A}_{\ell+1}, u \in \mathcal{A}_\ell \mid \mathcal{A}_\ell = a_\ell\right) \mathbb{P}\left(\mathcal{A}_\ell = a_\ell\right)$$
$$= \sum_{\ell=1}^{\ell_u^*-1} \sum_{\substack{a_\ell \subseteq [K]:\\ u \in a_\ell, \\ |a_\ell| = K2^{-\ell+1}}} \mathbb{P}\left(\exists A \subset \mathcal{A}_\ell, \text{ s.t. } |A| = \frac{|\mathcal{A}_\ell|}{2}, \forall i \in A, \, \hat{\mu}_u \leq \hat{\mu}_i \mid \mathcal{A}_\ell = a_\ell\right) \mathbb{P}\left(\mathcal{A}_\ell = a_\ell\right)$$

$$=\sum_{\ell=1}^{\ell_u^*-1}\sum_{\substack{a_\ell \subseteq [K]:\\ u \in a_\ell,\\ |a_\ell| = K2^{-\ell+1}}} \mathbb{P}\left(\exists A \subset a_\ell, \text{ s.t. } |A| = \frac{|a_\ell|}{2}, \forall i \in A, \, \hat{\mu}_u \le \hat{\mu}_i \mid \mathcal{A}_\ell = a_\ell\right) \mathbb{P}\left(\mathcal{A}_\ell = a_\ell\right)$$

Let $A \subset a_{\ell}$ be a set as described in $\mathbb{P}(\cdot)$ above. We denote by $\mathsf{Bot}_j(A)$ a set of arms with |A| - j lowest means in A. Formally, $\mathsf{Bot}_j(A)$ satisfies that $\mathsf{Bot}_j(A) \subseteq A$, $|\mathsf{Bot}_j(A)| = |A| - j$, and $\forall x \in \mathsf{Bot}_j(A)$, $\forall y \in A \setminus \mathsf{Bot}_j(A)$, $\mu_x \leq \mu_y$. We denote $\mu_i(A)$ to be mean of the arm *i* indexed within the set A. Also to make it clear, we denote $\mu_i([K])$ to be mean of the arm *i* indexed within the set [K], i.e., $\mu_i([K]) = \mu_i$

We set $j = \frac{|A|}{2} = \frac{|a_{\ell}|}{4}$, we obtain the following properties

- $\left|\operatorname{Bot}_{j}(A)\right| = \frac{|A|}{2} = \frac{|a_{\ell}|}{4}.$
- $\forall i \in \mathsf{Bot}_j(A), \mu_i \leq \mu_{|A| / 2}(A) \leq \mu_{|A| / 2}([K]) = \mu_{|a_\ell| / 4}([K]) \leq \mu_{|a_\ell| / 4}$. The first inequality is from the setup of $\mathsf{Bot}_j(A)$. The second inequality uses the fact that if $A \subseteq [K]$, hence $\forall i, \mu_i(A) \leq \mu_i([K])$

Within stage $\ell \leq \ell_u^* - 1$ and set $a_\ell \in [K]$ such that $u \in a_\ell$ and $|a_\ell| = K2^{-\ell+1}$, we focus on the probability

$$\mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in A, \, \hat{\mu}_{u} \leq \hat{\mu}_{i} \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$

$$\leq \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \mathsf{Bot}_{j}(A), \, \hat{\mu}_{u} \leq \hat{\mu}_{i} \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$
(use $\mathsf{Bot}_{j}(A) \subset A$)

Within the condition $\exists A \subset a_{\ell}$, s.t. $|A| = \frac{|a_{\ell}|}{2}$, $\forall i \in \mathsf{Bot}_j(A)$, we have $\hat{\mu}_i \leq \hat{\mu}_i$

$$\begin{aligned} & \Rightarrow \hat{\mu}_{u} \leq \mu_{i} \\ \Leftrightarrow \hat{\mu}_{u} - \mu_{u} - \hat{\mu}_{i} + \mu_{i} \leq -\mu_{u} + \mu_{i} \\ \Rightarrow \hat{\mu}_{u} - \mu_{u} - \hat{\mu}_{i} + \mu_{i} \leq -\mu_{u} + \mu_{|\frac{a_{\ell}|}{4}} \end{aligned} \qquad (\text{use the property } \mu_{i} \leq \mu_{|\frac{a_{\ell}|}{4}}) \\ \Leftrightarrow \hat{\mu}_{u} - \mu_{u} - \hat{\mu}_{i} + \mu_{i} \leq \Delta_{u} - \Delta_{|\frac{a_{\ell}|}{4}} \\ \Rightarrow \hat{\mu}_{u} - \mu_{u} - \hat{\mu}_{i} + \mu_{i} \leq \frac{1}{2} \Delta_{\frac{K}{4}2^{-\ell_{u}^{*}+1}} - \Delta_{|\frac{a_{\ell}|}{4}} \end{aligned} \qquad (\text{by the definition } \Delta_{u} \leq \frac{1}{2} \Delta_{\frac{K}{4}2^{-\ell_{u}^{*}+1}}) \\ \Rightarrow \hat{\mu}_{u} - \mu_{u} - \hat{\mu}_{i} + \mu_{i} \leq \frac{1}{2} \Delta_{\frac{K}{4}2^{-\ell_{u}^{*}+1}} - \Delta_{|\frac{a_{\ell}|}{4}} \end{aligned} \qquad (\text{since } \ell < \ell_{u}^{*}) \\ \Leftrightarrow \hat{\mu}_{u} - \mu_{u} - \hat{\mu}_{i} + \mu_{i} \leq \frac{1}{2} \Delta_{|\frac{a_{\ell}|}{4}} - \Delta_{|\frac{a_{\ell}|}{4}} \end{aligned}$$

Thus, we have

$$\mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \mathsf{Bot}_{j}(A), \hat{\mu}_{u} \leq \hat{\mu}_{i} \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$

$$\leq \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \mathsf{Bot}_{j}(A), \hat{\mu}_{u} - \mu_{u} - \hat{\mu}_{i} + \mu_{i} \leq -\frac{1}{2}\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor} \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$

$$\leq \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \mathsf{Bot}_{j}(A), \left(\hat{\mu}_{u} - \mu_{u} \leq -\frac{1}{4}\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}\right) \vee \left(\hat{\mu}_{i} - \mu_{i} \geq \frac{1}{4}\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}\right) \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$

$$\leq \mathbb{P}\left(\hat{\mu}_{u} - \mu_{u} \leq -\frac{1}{4}\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor} \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$

$$+ \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \mathsf{Bot}_{j}(A), \hat{\mu}_{i} - \mu_{i} \geq \frac{1}{4}\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor} \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$

$$(\text{use union bound})$$

$$\leq \mathbb{P}\left(\hat{\mu}_{u} - \mu_{u} \leq -\frac{1}{4}\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor} \mid \mathcal{A}_{\ell} = a_{\ell}\right)$$

$$\begin{split} &+ \sum_{\substack{A \subset a_{\ell}:\\|A| = \frac{|a_{\ell}|}{2}}} \mathbb{P}\left(\forall i \in \mathsf{Bot}_{j}(A), \hat{\mu}_{i} \geq \mu_{i} + \frac{1}{4} \Delta_{\lfloor \frac{|a_{\ell}|}{4}} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \\ &\leq \exp\left(-\frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + \sum_{\substack{A \subset a_{\ell}:\\|A| = \frac{|a_{\ell}|}{2}}} \exp\left(-|\mathsf{Bot}_{j}(A)| \frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) \qquad \text{(use Hoeffding's inequality)} \\ &= \exp\left(-\frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + \sum_{\substack{A \subset a_{\ell}:\\|A| = \frac{|a_{\ell}|}{2}}} \exp\left(-\frac{|a_{\ell}|}{4} \frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) \right) \qquad \text{(use property } |\mathsf{Bot}_{j}(A)| = \frac{|a_{\ell}|}{4}) \\ &= \exp\left(-\frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + \binom{|a_{\ell}|}{2} \exp\left(-\frac{|a_{\ell}|}{4} \frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) \\ &\leq \exp\left(-\frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + (2e)^{\frac{|a_{\ell}|}{2}} \exp\left(-\frac{|a_{\ell}|}{4} \frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) \\ &= \exp\left(-\frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + \exp\left(-\frac{|a_{\ell}|}{4} \frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) \right) \\ &= \exp\left(-\frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + \exp\left(-\frac{|a_{\ell}|}{4} \frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) \right) \\ &= \exp\left(-\frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + \exp\left(-\frac{|a_{\ell}|}{4} \frac{N(m,\ell)}{2} \frac{\Delta_{\lfloor \frac{|a_{\ell}|}{4}}}{16}\right) + \frac{|a_{\ell}|}{2}\log(2e)\right). \end{split}$$

We apply Lemma 23 that shows for all phase m such that $T_m \ge T_{\delta}$, $N^{(m,\ell)} \ge \frac{256}{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^2} \log (2e)$. This inequality also means that

$$\begin{split} \frac{256}{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}} \log \left(2e \right) &\leq N^{(m,\ell)} \\ \Leftrightarrow 8 \log \left(2e \right) &\leq \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{16} \\ \Leftrightarrow |a_{\ell}| \log \left(2e \right) &\leq \frac{|a_{\ell}|}{8} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{16} \\ \Rightarrow \frac{|a_{\ell}|}{2} \log \left(2e \right) &\leq \frac{|a_{\ell}|}{8} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{16} \\ \Rightarrow -\frac{|a_{\ell}|}{4} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{16} + \frac{|a_{\ell}|}{2} \log (2e) \leq -\frac{|a_{\ell}|}{8} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{16}. \end{split}$$

Thus,

$$\exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{|a_{\ell}|}^{2}}{16}\right) + \exp\left(-\frac{|a_{\ell}|}{4}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{|a_{\ell}|}^{2}}{16} + \frac{|a_{\ell}|}{2}\log(2e)\right)$$

$$\leq \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{|a_{\ell}|}^{2}}{16}\right) + \exp\left(-\frac{|a_{\ell}|}{8}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{|a_{\ell}|}^{2}}{16}\right)$$

$$\leq \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{|a_{\ell}|}^{2}}{16}\right) + \exp\left(-\frac{1}{8}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{|a_{\ell}|}^{2}}{16}\right)$$
(use $|a_{\ell}| \ge 1$)

$$\leq 2 \exp\left(-\frac{\Delta_{\lfloor a_{\ell} \rfloor}^{2} N^{(m,\ell)}}{256}\right)$$
$$= 2 \exp\left(-\frac{\Delta_{\frac{K}{4}2^{-\ell+1}}^{2} N^{(m,\ell)}}{256}\right) \qquad (\text{use } |a_{\ell}| = K2^{-\ell+1})$$

By the definition $N^{(m,\ell)} = \frac{T_m}{2^{-\ell+1}K \log_2(K)}$, we further have $\int \Delta^2_{K 2^{-\ell+1}} T$

$$\exp\left(-\frac{\Delta_{\frac{K}{4}2^{-\ell+1}}^{2}}{256} \cdot \frac{T_{m}}{2^{-\ell+1}K\log_{2}(K)}\right)$$

$$= 2\exp\left(-\frac{\Delta_{\frac{K}{4}2^{-\ell+1}}^{2}}{\frac{K}{4}2^{-\ell+1}} \cdot \frac{T_{m}}{1024\log_{2}(K)}\right)$$

$$\leq 2\exp\left(-\frac{1}{H_{2}} \cdot \frac{T_{m}}{1024\log_{2}(K)}\right) \qquad \text{(by the definition } H_{2} = \max_{i} i\Delta_{i}^{-2})$$

$$= 2\exp\left(-\frac{T_{m}}{1024H_{2}\log_{2}(K)}\right).$$

To summarize, we have obtained the following probability bound

$$\mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in A, \, \hat{\mu}_u \leq \hat{\mu}_i \mid \mathcal{A}_{\ell} = a_{\ell}\right) \leq 2\exp\left(-\frac{T_m}{1024H_2\log_2(K)}\right).$$

Therefore, from the beginning derivation, we obtain the following probability bound

$$\mathbb{P}\left(\ell_{u} < \ell_{u}^{*}\right) = \sum_{\ell=1}^{\ell_{u}^{*}-1} \sum_{\substack{a_{\ell} \subseteq [K]:\\u \in a_{\ell},\\|a_{\ell}| = K \cdot 2^{-\ell+1}}} \mathbb{P}\left(u \notin \mathcal{A}_{\ell+1}, u \in \mathcal{A}_{\ell} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right) \\
= \sum_{\ell=1}^{\ell_{u}^{*}-1} \sum_{\substack{a_{\ell} \subseteq [K]:\\u \in a_{\ell},\\|a_{\ell}| = K \cdot 2^{-\ell+1}}} 2 \exp\left(-\frac{T_{m}}{1024H_{2}\log_{2}(K)}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right) \\
= \sum_{\ell=1}^{\ell_{u}^{*}-1} 2 \exp\left(-\frac{T_{m}}{1024H_{2}\log_{2}(K)}\right) \\
\leq 2 \log_{2}(K) \exp\left(-\frac{T_{m}}{1024H_{2}\log_{2}(K)}\right) \qquad (\text{use } \ell_{u}^{*} \leq \log_{2}(K)) \\
= \exp\left(-\frac{T_{m}}{1024H_{2}\log_{2}(K)} + \log(2\log_{2}(K))\right).$$

The condition $T_m \ge T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$ implies that

$$T_{m} \geq 4096H_{2}\log_{2}(K)\log\left(\frac{6K\log_{2}(K)}{\delta}\right)$$

$$\Rightarrow T_{m} \geq 2048H_{2}\log_{2}(K)\log\left(2\log_{2}(K)\right) \qquad (\text{use } \delta \leq 1)$$

$$\Rightarrow \log\left(2\log_{2}(K)\right) \leq \frac{T_{m}}{2048H_{2}\log_{2}(K)}$$

$$\Rightarrow -\frac{T_{m}}{1024H_{2}\log_{2}(K)} + \log\left(2\log_{2}(K)\right) \leq -\frac{T_{m}}{2048H_{2}\log_{2}(K)}.$$

Therefore,

$$\begin{split} & \exp\left(-\frac{T_m}{1024H_2\log_2(K)} + \log(2\log_2(K))\right) \\ & \leq \exp\left(-\frac{T_m}{2048H_2\log_2(K)}\right), \end{split}$$

which concludes the proof of the Lemma.

Lemma 20. For all phase m such that $T_m \ge T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$, the probability that FC-DSH fails to output arm 1 at the end of phase m satisfies

$$\mathbb{P}\left(J_m \neq 1\right) \le \exp\left(-\frac{T_m}{512H_2\log_2(K)}\right).$$

Proof. While one can directly use the proof of Karnin et al. (2013a), we present here an alternative proof that has an pedagogical value – this proof shows a more fine-grained control of events that reveals that the bottleneck of the guarantee is the bad behavior of the best arm rather than the bad behavior of the bad arms.

We denote by \mathcal{A}_{ℓ} the set of surviving arms at stage ℓ . We denote by ℓ_1 the stage at which arm 1 is eliminated. The event that $J_m \neq 1$ implies that arm 1 does not survive until the last stage L, i.e., $\ell_1 < L$. We have

$$\begin{split} & \mathbb{P}\left(\ell_{1} < L\right) \\ &= \mathbb{P}\left(\exists \ell \leq L - 1 : 1 \notin \mathcal{A}_{\ell+1}, 1 \in \mathcal{A}_{\ell}\right) \\ &\leq \sum_{\ell=1}^{L-1} \mathbb{P}\left(1 \notin \mathcal{A}_{\ell+1}, 1 \in \mathcal{A}_{\ell}\right) \qquad (\text{use union bound}) \\ &\leq \sum_{\ell=1}^{L-1} \sum_{\substack{a_{\ell} \subseteq [K]:\\1 \in a_{\ell},\\|a_{\ell}| = K \cdot 2^{-\ell+1}}} \mathbb{P}\left(1 \notin \mathcal{A}_{\ell+1}, 1 \in \mathcal{A}_{\ell}, \mathcal{A}_{\ell} = a_{\ell}\right) \qquad (\text{use the law of total probability}) \\ &= \sum_{\ell=1}^{L-1} \sum_{\substack{a_{\ell} \subseteq [K]:\\1 \in a_{\ell},\\|a_{\ell}| = K \cdot 2^{-\ell+1}}} \mathbb{P}\left(1 \notin \mathcal{A}_{\ell+1}, 1 \in \mathcal{A}_{\ell} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right). \end{split}$$

By the setup of the FC-DSH, at stage ℓ , the event where arm 1 is in set \mathcal{A}_{ℓ} but is eliminated next around means that there exists a set $A \subset \mathcal{A}_{\ell}$ that half the size of \mathcal{A}_{ℓ} such that $\forall i \in A, \hat{\mu}_1 \leq \hat{\mu}_i$. Formally,

$$\begin{split} &\sum_{\ell=1}^{L-1} \sum_{\substack{a_{\ell} \subseteq [K]:\\ 1 \in a_{\ell},\\ |a_{\ell}| = K2^{-\ell+1}}} \mathbb{P}\left(u \notin \mathcal{A}_{\ell+1}, 1 \in \mathcal{A}_{\ell} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right) \\ &= \sum_{\ell=1}^{L-1} \sum_{\substack{a_{\ell} \subseteq [K]:\\ 1 \in a_{\ell},\\ |a_{\ell}| = K2^{-\ell+1}}} \mathbb{P}\left(\exists A \subset \mathcal{A}_{\ell}, \text{ s.t. } |A| = \frac{|\mathcal{A}_{\ell}|}{2}, \forall i \in A, \, \hat{\mu}_{1} \leq \hat{\mu}_{i} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right) \\ &= \sum_{\ell=1}^{L-1} \sum_{\substack{a_{\ell} \subseteq [K]:\\ 1 \in a_{\ell},\\ |a_{\ell}| = K2^{-\ell+1}}} \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in A, \, \hat{\mu}_{1} \leq \hat{\mu}_{i} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right). \end{split}$$

Let $A \subset a_{\ell}$ be a set as described inside $\mathbb{P}(\cdot)$ above. We denote by $\mathsf{Bot}_j(A)$ be a set of arms with |A| - j lowest means in A. Formally, $\mathsf{Bot}_j(A)$ satisfies that $\mathsf{Bot}_j(A) \subseteq A$, $|\mathsf{Bot}_j(A)| = |A| - j$, and $\forall x \in \mathsf{Bot}_j(A)$, $\forall y \in A \setminus \mathsf{Bot}_j(A)$, $\mu_x \leq \mu_y$.

We denote $\mu_i(A)$ to be mean of the arm i indexed within the set A. Also to make it clear, we denote $\mu_i([K])$ to

be mean of the arm i indexed within the whole set [K], i.e., $\mu_i([K]) = \mu_i$. We set $j = \frac{|A|}{2} = \frac{|a_\ell|}{4}$, we obtain the following properties

- $\left|\operatorname{Bot}_{j}(A)\right| = \frac{|A|}{2} = \frac{|a_{\ell}|}{4}.$
- $\forall i \in \mathsf{Bot}_j(A), \ \mu_i \leq \mu_{\lfloor A \rfloor \over 2}(A) \leq \mu_{\lfloor A \rfloor \over 2}([K]) = \mu_{\lfloor a_\ell \rfloor \over 4}([K]) \leq \mu_{\lfloor a_\ell \rfloor \over 4}$. The first inequality is from the definition of $\mathsf{Bot}_j(A)$. The second inequality uses the fact that if $A \subseteq [K]$, hence $\forall i, \ \mu_i(A) \leq \mu_i([K])$

Within stage $\ell \leq L-1$ and set $a_{\ell} \subseteq [K], 1 \in a_{\ell}, |a_{\ell}| = K2^{-\ell+1}$, we focus on the probability

$$\begin{split} &= \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \text{Bot}_{j}(A), \hat{\mu}_{1} - \mu_{1} - \hat{\mu}_{i} + \mu_{i} \leq -\Delta_{\frac{|a_{\ell}|}{4}} | \mathcal{A}_{\ell} = a_{\ell}\right) \\ &\leq \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \text{Bot}_{j}(A), \left(\hat{\mu}_{1} - \mu_{1} \leq -\frac{1}{2}\Delta_{\frac{|a_{\ell}|}{4}}\right) \lor \left(\hat{\mu}_{i} - \mu_{i} \geq \frac{1}{2}\Delta_{\frac{|a_{\ell}|}{4}}\right) | \mathcal{A}_{\ell} = a_{\ell}\right) \\ &+ \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \text{Bot}_{j}(A), \hat{\mu}_{i} - \mu_{i} \geq \frac{1}{2}\Delta_{\frac{|a_{\ell}|}{4}} | \mathcal{A}_{\ell} = a_{\ell}\right) \\ &+ \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \text{Bot}_{j}(A), \hat{\mu}_{i} - \mu_{i} \geq \frac{1}{2}\Delta_{\frac{|a_{\ell}|}{4}} | \mathcal{A}_{\ell} = a_{\ell}\right) \\ &+ \mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in \text{Bot}_{j}(A), \hat{\mu}_{i} - \mu_{i} \geq \frac{1}{2}\Delta_{\frac{|a_{\ell}|}{4}} | \mathcal{A}_{\ell} = a_{\ell}\right) \\ &+ \sum_{\substack{A \subset a_{\ell}: \\ |A| = \frac{|a_{\ell}|}{4}}} \mathbb{P}\left(\forall i \in \text{Bot}_{j}(A), \hat{\mu}_{i} \geq \mu_{i} + \frac{1}{2}\Delta_{\frac{|a_{\ell}|}{4}} | \mathcal{A}_{\ell} = a_{\ell}\right) \\ &\leq \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{\frac{|a_{\ell}|}{4}}}{4}\right) + \sum_{\substack{A \subset a_{\ell}: \\ |A| = \frac{|a_{\ell}|}{2}}} \exp\left(-\left|\text{Bot}_{j}(A)\right| \frac{N^{(m,\ell)}}{2}\frac{\Delta_{\frac{|a_{\ell}|}}^{2}}{4}\right) \right) \qquad (\text{use Hoeffding's inequality}) \\ &= \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{\frac{|a_{\ell}|}}^{2}}{4}\right) + \binom{|a_{\ell}|}{2}\exp\left(-\frac{|a_{\ell}|}{4}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{\frac{|a_{\ell}|}}^{2}}{4}\right) \\ &\leq \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{\frac{|a_{\ell}|}}^{2}}{4}\right) + \binom{|a_{\ell}|}{2}\exp\left(-\frac{|a_{\ell}|}{4}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{\frac{|a_{\ell}|}}^{2}}{4}\right) \qquad (\text{use Inemma 24}) \end{aligned}$$

$$= \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{\lfloor \underline{a_{\ell}} \rfloor}^{2}}{4}\right) + \exp\left(-\frac{|a_{\ell}|}{4}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{\lfloor \underline{a_{\ell}} \rfloor}^{2}}{4} + \frac{|a_{\ell}|}{2}\log(2e)\right).$$

We apply Lemma 23 that shows for all phase m such that $T_m \ge T_{\delta}$, $N^{(m,\ell)} \ge \frac{256}{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^2} \log (2e)$. This inequality also means that

$$\begin{split} \frac{256}{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}} \log (2e) &\leq N^{(m,\ell)} \\ \Leftrightarrow 32 \log (2e) &\leq \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{4} \\ \Leftrightarrow 4|a_{\ell}| \log (2e) &\leq \frac{|a_{\ell}|}{8} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{4} \\ \Rightarrow \frac{|a_{\ell}|}{2} \log (2e) &\leq \frac{|a_{\ell}|}{8} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{4} \\ \Rightarrow -\frac{|a_{\ell}|}{4} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{4} + \frac{|a_{\ell}|}{2} \log (2e) \leq -\frac{|a_{\ell}|}{8} \frac{N^{(m,\ell)}}{2} \frac{\Delta_{\lfloor \frac{a_{\ell}}{4} \rfloor}^{2}}{4}. \end{split}$$

Thus,

$$\begin{split} \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}}{4}\right) + \exp\left(-\frac{|a_{\ell}|}{4}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}}{4} + \frac{|a_{\ell}|}{2}\log(2e)\right) \\ &\leq \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}}{4}\right) + \exp\left(-\frac{|a_{\ell}|}{8}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}}{4}\right) \\ &\leq \exp\left(-\frac{N^{(m,\ell)}}{2}\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}}{4}\right) + \exp\left(-\frac{1}{8}\frac{N^{(m,\ell)}}{2}\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}}{4}\right) \qquad (\text{use } |a_{\ell}| \ge 1) \\ &\leq 2\exp\left(-\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}N^{(m,\ell)}}{64}\right) \\ &= 2\exp\left(-\frac{\Delta_{[\frac{a_{\ell}}{4}]}^{2}N^{(m,\ell)}}{64}\right) \qquad (\text{use } |a_{\ell}| = K2^{-\ell+1}) \end{split}$$

By the definition $N^{(m,\ell)} = \frac{T_m}{2^{-\ell+1}K \log_2(K)}$, we further have

$$\begin{split} \exp\left(-\frac{\Delta_{\frac{K}{4}2^{-\ell+1}}^{2}}{64} \cdot \frac{T_{m}}{2^{-\ell+1}K \log_{2}(K)}\right) \\ &= 2 \exp\left(-\frac{\Delta_{\frac{K}{4}2^{-\ell+1}}^{2}}{\frac{K}{4}2^{-\ell+1}} \cdot \frac{T_{m}}{256 \log_{2}(K)}\right) \\ &\leq 2 \exp\left(-\frac{1}{H_{2}} \cdot \frac{T_{m}}{256 \log_{2}(K)}\right) \\ &= 2 \exp\left(-\frac{T_{m}}{256 H_{2} \log_{2}(K)}\right). \end{split}$$

(by the definition $H_2 = \max_i i \Delta_i^{-2}$)

To summarize, we have obtained the following probability bound

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$$\mathbb{P}\left(\exists A \subset a_{\ell}, \text{ s.t. } |A| = \frac{|a_{\ell}|}{2}, \forall i \in A, \, \hat{\mu}_1 \leq \hat{\mu}_i \mid \mathcal{A}_{\ell} = a_{\ell}\right) \leq 2\exp\left(-\frac{T_m}{256H_2\log_2(K)}\right).$$

Therefore, from the beginning derivation, we obtain the following probability bound

$$\begin{split} & \mathbb{P}\left(\ell_{u} < L\right) \\ &= \sum_{\ell=1}^{L-1} \sum_{\substack{a_{\ell} \subseteq [K]: \\ 1 \in a_{\ell}, \\ |a_{\ell}| = K \cdot 2^{-\ell+1}}} \mathbb{P}\left(u \notin \mathcal{A}_{\ell+1}, u \in \mathcal{A}_{\ell} \mid \mathcal{A}_{\ell} = a_{\ell}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right) \\ &= \sum_{\ell=1}^{L-1} \sum_{\substack{a_{\ell} \subseteq [K]: \\ 1 \in a_{\ell}, \\ |a_{\ell}| = K \cdot 2^{-\ell+1}}} 2 \exp\left(-\frac{T_{m}}{256H_{2}\log_{2}(K)}\right) \mathbb{P}\left(\mathcal{A}_{\ell} = a_{\ell}\right) \\ &= \sum_{\ell=1}^{L-1} 2 \exp\left(-\frac{T_{m}}{256H_{2}\log_{2}(K)}\right) \\ &\leq 2 \log_{2}(K) \exp\left(-\frac{T_{m}}{256H_{2}\log_{2}(K)}\right) \\ &= \exp\left(-\frac{T_{m}}{256H_{2}\log_{2}(K)} + \log(2\log_{2}(K))\right). \end{split}$$
(use $L = \log_{2}(K)$)

The condition $T_m \ge T_{\delta} = 4096 H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$ implies that

$$T_m \ge 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$$

$$\Rightarrow T_m \ge 2048H_2 \log_2(K) \log\left(2 \log_2(K)\right)$$

$$\Rightarrow T_m \ge 512H_2 \log_2(K) \log\left(2 \log_2(K)\right)$$

$$\Rightarrow \log\left(2 \log_2(K)\right) \le \frac{T_m}{512H_2 \log_2(K)}$$

$$\Rightarrow -\frac{T_m}{256H_2 \log_2(K)} + \log\left(2 \log_2(K)\right) \le -\frac{T_m}{512H_2 \log_2(K)}.$$

$$\exp\left(-\frac{T_m}{256H_2 \log_2(K)} + \log\left(2 \log_2(K)\right)\right) \le -\frac{T_m}{512H_2 \log_2(K)}.$$

Therefore,

$$\exp\left(-\frac{T_m}{256H_2\log_2(K)} + \log(2\log_2(K))\right)$$
$$\leq \exp\left(-\frac{T_m}{512H_2\log_2(K)}\right),$$

which concludes the proof of the Lemma.

Lemma 21. Let $u \neq 1$ be any non-optimal that satisfies $\Delta_u > \frac{1}{2}\Delta_{\frac{K}{4}}$. For all phase m such that $T_m \geq T_{\delta} =$ $4096H_2\log_2(K)\log\left(\frac{6K\log_2(K)}{\delta}\right)$, the number of samples used at stage ℓ_u when arm u is eliminated, satisfies

$$N^{(m,\ell_u)} \ge \frac{32}{\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$$

Proof. We start from the condition $T_m \ge 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$ and obtain the inequality as follows

$$T_m \ge 4096H_2\log_2(K)\log\left(\frac{6K\log_2(K)m^2}{\delta}\right)$$

$$\Rightarrow T_m \ge 1024 \frac{\frac{K}{4}}{\Delta_{\frac{K}{4}}^2} \log_2(K) \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$$
 (by the definition $H_2 = \max_{i\ge 2} i\Delta_i^{-2}$)

$$\Rightarrow T_m \ge 1024 \frac{\frac{K}{4}2^{-\ell_u}}{\Delta_{\frac{K}{4}}^2} \log_2(K) \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$$
 (use $1 \ge 2^{-\ell_u}$)

$$\Rightarrow \frac{T_m}{K2^{-\ell_u+1}\log_2(K)} \ge \frac{128}{\Delta_{\frac{K}{4}}^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$$
 (by the definition of $N^{(m,\ell)}$)

$$\Rightarrow N^{(m,\ell_u)} \ge \frac{128}{4\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$$
 (by the definition of $N^{(m,\ell)}$)

$$\Rightarrow N^{(m,\ell_u)} \ge \frac{128}{4\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$$
 (use $\Delta_u > \frac{1}{2}\Delta_{\frac{K}{4}}$)

$$\Rightarrow N^{(m,\ell_u)} \ge \frac{32}{\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right).$$

Lemma 22. Let $u \neq 1$ be any non-optimal arm. For all phase m such that $T_m \geq T_{\delta} = 4096H_2\log_2(K)\log\left(\frac{6K\log_2(K)}{\delta}\right)$, assuming ℓ_u^* exists, the number of samples used at stage ℓ_u^* satisfies

$$N^{(m,\ell_u^*)} \ge \frac{32}{\Delta_u^2} \log\left(\frac{6K \log_2(K)m^2}{\delta}\right)$$

Proof. Recall that ℓ_u^* is defined to be the largest stage ℓ such that $\Delta_u \leq \frac{1}{2}\Delta_{\frac{K}{4}\cdot 2^{-\ell+1}}$. For any arm $u \in [2, K]$, since it survives until stage ℓ_u^* , by the definition, it satisfies $\frac{1}{2}\Delta_{\frac{K}{4}\cdot 2^{-\ell_u^*}} \leq \Delta_i \leq \frac{1}{2}\Delta_{\frac{K}{4}\cdot 2^{-\ell_u^*+1}}$.

We start from the condition $T_m \ge 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$ and obtain the inequality as follows

$$T_{m} \geq 4096H_{2} \log_{2}(K) \log\left(\frac{6K \log_{2}(K)m^{2}}{\delta}\right)$$

$$\Rightarrow T_{m} \geq 1024 \frac{\frac{K}{4}2^{-\ell_{u}^{*}}}{\Delta_{\frac{K}{4}\cdot2^{-\ell_{u}^{*}}}^{2}} \log_{2}(K) \log\left(\frac{6K \log_{2}(K)m^{2}}{\delta}\right) \qquad \text{(by the definition } H_{2} = \max_{i \geq 2} i\Delta_{i}^{-2})$$

$$\Leftrightarrow \frac{T_{m}}{K2^{-\ell_{u}^{*}+1} \log_{2}(K)} \geq \frac{128}{\Delta_{\frac{K}{4}\cdot2^{-\ell_{u}^{*}}}^{2}} \log\left(\frac{6K \log_{2}(K)m^{2}}{\delta}\right) \qquad \text{(by the definition of } N^{(m,\ell)})$$

$$\Rightarrow N^{(m,\ell_{u}^{*})} \geq \frac{128}{4\Delta_{u}^{2}} \log\left(\frac{6K \log_{2}(K)m^{2}}{\delta}\right) \qquad \text{(use } \Delta_{u} \geq \frac{1}{2}\Delta_{\frac{K}{4}\cdot2^{-\ell_{u}^{*}}}^{2})$$

$$\Leftrightarrow N^{(m,\ell_{u}^{*})} \geq \frac{32}{\Delta_{u}^{2}} \log\left(\frac{6K \log_{2}(K)m^{2}}{\delta}\right).$$

Lemma 23. For all phase m such that $T_m \ge T_{\delta} = 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right)$, the number of samples used at stage ℓ where $|a_{\ell}| = K2^{-\ell+1}$, satisfies

$$N^{(m,\ell)} \ge \frac{256}{\Delta_{\frac{|a_\ell|}{4}}^2} \log\left(2e\right).$$

Proof. We start from the condition $T_m \ge 4096H_2\log_2(K)\log\left(\frac{6K\log_2(K)}{\delta}\right)$ and obtain the inequality as follows

$$\begin{split} T_m &\geq 4096H_2 \log_2(K) \log\left(\frac{6K \log_2(K)}{\delta}\right) \\ \Rightarrow T_m &\geq 1024H_2 \log_2(K) \log\left(2e\right) \qquad (\text{since } \delta \leq 1) \\ \Rightarrow T_m &\geq 1024 \frac{\frac{K}{4}2^{-\ell+1}}{\Delta_{\frac{K}{4}2^{-\ell+1}}^2} \log_2(K) \log\left(2e\right) \qquad (\text{by the definition } H_2 = \max_{i\geq 2} i\Delta_i^{-2}) \\ \Leftrightarrow \frac{T_m}{K2^{-\ell+1} \log_2(K)} &\geq \frac{256}{\Delta_{\frac{K}{4}2^{-\ell+1}}^2} \log\left(2e\right) \\ \Leftrightarrow N^{(m,\ell)} &\geq \frac{256}{\Delta_{\frac{K}{4}2^{-\ell+1}}^2} \log\left(2e\right) \qquad (\text{by the definition of } N^{(m,\ell)}) \\ \Leftrightarrow N^{(m,\ell)} &\geq \frac{256}{\Delta_{\frac{L}{4}2}^2}^2 \log\left(2e\right) . \qquad (a_\ell) = K2^{-\ell+1}) \end{split}$$

Lemma 24 (Stirling's formula (Das, 2016)). Let k, d be two positive integers such that $1 \le k \le n$, then,

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

Lemma 25. ,

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

C BrakeBooster

C.1 Proof of Theorem 12

We prove a slightly more general version of Theorem 12.

Theorem 26 (Exponential tail; full version). Let an algorithm \mathcal{A} be δ -correct and have a high probability sample complexity of $T^*_{\delta}(\mathcal{A})$. Suppose we run BrakeBooster (Algorithm 2) with input \mathcal{A} , $L_1 = \lceil \frac{4 \log(1+\frac{2}{\delta})}{\log \frac{1}{4e\delta_0}} \rceil$, $T_1 \ge 1$, and $\delta_0 \le (\frac{1}{2e})^2$. Let $T_0 = 24\bar{T}^* \log_2^2(\frac{16\bar{T}^*}{T_1})L_1$ where $\bar{T}^* = T^*_{\delta_0}(\mathcal{A}) \lor T_1$. Then, $\forall T \ge T_0, \ \mathbb{P}\left(\tau(\mathcal{M}) \ge T\right) \le \exp\left(-\frac{T}{768T^*_{\delta_0}(\mathcal{A})\log_2 T}\ln\frac{1}{\delta_0}\right)$

Proof. Let $r^* := \min\{r \in \mathbb{N}_+ : T_{r,r} \ge T^*_{\delta_0}(\mathcal{A})\}$. Let $\overline{T}_{r,c}$ be the total amount samples consumed up to and including stage (r,c). Define $R_{r,c} := \{\ell \in [L_{r,c}] : \ell$ -th trial does not self-terminate $\}$.

Let $r > r^*$. Then,

$$\mathbb{P}\left(\tau(\mathcal{M}) \geq \bar{T}_{r,c}\right) \leq \mathbb{P}(J_{r-1,r^*} = 0)$$

Note that the fact that a stage returns 0 implies that more than a half of the trials did not self-terminate. Thus

$$\mathbb{P}(J_{r-1,r^*} = 0) \le \mathbb{P}\left(\exists \text{ at least } (\lfloor L_{r-1,r^*}/2 \rfloor + 1) \text{ trials that do not self-terminate }\right)$$

$$\leq \mathbb{P}\left(|R_{r-1,r^*}| \geq \frac{L_{r-1,r^*}}{2}\right)$$

Since the trials in stage $(r-1, r^*)$ use samples more than $T^*_{\delta_0}(\mathcal{A})$, by the sample complexity guarantee of \mathcal{A} and Lemma 28, we have

$$\mathbb{P}\left(|R_{r-1,r^*}| \ge \frac{L_{r-1,r^*}}{2}\right) \le \exp\left(-\frac{L_{r-1,r^*}}{2}\log\frac{1}{2e\delta_0}\right) \\
\le \exp\left(-\frac{2^{r-1-r^*}(r-1)L_1}{2}\ln\sqrt{\frac{1}{\delta_0}}\right) \qquad (\sqrt{\delta_0} \le \frac{1}{2e}) \\
= \exp\left(-\frac{2^{r-1-r^*}(r-1)L_1}{4}\ln\frac{1}{\delta_0}\right)$$

Note that by Lemma 30, we have, $\forall r \geq 2$,

$$\bar{T}_{r,c} \ge \frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2 \ge T_1 L_1 2^r \implies r \le \log_2\left(\frac{\bar{T}_{r,c}}{L_1 T_1}\right)$$

Then,

$$\mathbb{P}\left(\tau(\mathcal{M}) \geq \bar{T}_{r,c}\right) = \exp\left(-\frac{2^{r-r^*-1}L_1}{4}(r-1)\ln\frac{1}{\delta_0}\right) \\
\leq \exp\left(-\frac{2^{r-r^*-1}L_1}{8}r\ln\frac{1}{\delta_0}\right) \qquad (r \geq 2 \implies r-1 \geq \frac{r}{2}) \\
= \exp\left(-\frac{2^{r-1}L_1}{8 \cdot 2^{r^*}}r\ln\frac{1}{\delta_0}\right) \\
= \exp\left(-\frac{2^{r-1}T_1L_1r^2}{8T_1r \cdot 2^{r^*}}\ln\frac{1}{\delta_0}\right) \\
\leq \exp\left(-\frac{\bar{T}_{r,c}/3}{8T_1r \cdot 2^{r^*}}\ln\frac{1}{\delta_0}\right) \qquad (\text{Lemma 30}) \\
\leq \exp\left(-\frac{\bar{T}_{r,c}/3}{8T_1r \cdot \frac{8\bar{T}_1^*}{T_1}}\ln\frac{1}{\delta_0}\right) \qquad (\text{Lemma 29}; \bar{T}^* := T^*_{\delta_0}(\mathcal{A}) \lor T_1)$$

$$= \exp\left(-\frac{\bar{T}_{r,c}/3}{64r \cdot \bar{T}^*} \ln \frac{1}{\delta_0}\right)$$
$$\leq \exp\left(-\frac{\bar{T}_{r,c}/3}{64\bar{T}^*\log_2 \bar{T}_{r,c}} \ln \frac{1}{\delta_0}\right) \qquad (r \ge 2, r \le \log_2 \frac{\bar{T}_{r,c}}{L_1 T_1} \le \log_2 \bar{T}_{r,c})$$

Hence,

$$\forall r > r^* \quad \forall c \le r \quad \mathbb{P}\left(\tau(\mathcal{M}) \ge \bar{T}_{r,c}\right) \le \exp\left(-\frac{\bar{T}_{r,c}}{192\bar{T}^*\log_2\bar{T}_{r,c}}\ln\frac{1}{\delta_0}\right)$$

Next, in order to obtain an exponential stopping tail guarantee, we need to upper bound $\mathbb{P}(\tau(\mathcal{M}) \geq T)$ for every T that is sufficiently large instead of those particular $\overline{T}_{r,c}$'s.

Let $T \geq \overline{T}_{r^*+1,1}$. We consider two cases.

Case 1. $T \in [\overline{T}_{r,c}, \overline{T}_{r,c+1})$ for some $r > r^*$ and $c \le r-1$. In this case, we have

$$\begin{split} \bar{T}_{r,c+1} &= \bar{T}_{r,c} + T_1 L_1 2^{r-1} \cdot r \\ \bar{T}_{r,c+1} &= 1 + \frac{T_1 L_1 2^{r-1} \cdot r}{\bar{T}_{r,c}} \\ &\leq 1 + \frac{T_1 L_1 2^{r-1} r}{\frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2} \\ &= 1 + \frac{2}{r} \\ \begin{pmatrix} d \\ \leq 2 \end{pmatrix} \qquad (r \geq r^* + 1 \geq 2) \end{split}$$

This implies $\bar{T}_{r,c} \leq T < 2\bar{T}_{r,c}$. Then,

$$\begin{aligned} \left(\tau(\mathcal{M}) \ge T\right) &\leq \mathbb{P}\left(\tau(\mathcal{M}) \ge \bar{T}_{r,c}\right) \\ &\leq \exp\left(-\frac{\bar{T}_{r,c}}{192\bar{T}^* \log_2 \bar{T}_{r,c}} \ln \frac{1}{\delta_0}\right) \\ &\leq \exp\left(-\frac{T}{384\bar{T}^* \log_2 T} \ln \frac{1}{\delta_0}\right) \end{aligned} \qquad (\bar{T}_{r,c} \le T < 2\bar{T}_{r,c}) \end{aligned}$$

Case 2. $T \in [\overline{T}_{r,r}, \overline{T}_{r+1,1})$ for some $r > r^*$. In this case,

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$$\begin{split} \bar{T}_{r+1,1} &= \bar{T}_{r,r} + T_1 L_1 2^r (r+1) \\ \bar{T}_{r+1,1} &= 1 + \frac{T_1 L_1 2^r (r+1)}{\bar{T}_{r,r}} \\ &\leq 1 + \frac{T_1 L_1 2^r (r+1)}{\frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2} \\ &= 1 + \frac{4(r+1)}{r^2} \\ &= 4 \end{split} \qquad (\bar{T}_{r,r} \geq \frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2)$$

Thus, we have $\bar{T}_{r,r} \leq T < 4\bar{T}_{r,r}$ and $\mathbb{P}\left(\tau(\mathcal{M}) \geq T\right) \leq \mathbb{P}\left(\tau(\mathcal{M}) \geq \bar{T}_{r,r}\right)$ $\leq \exp\left(-\frac{\bar{T}_{r,r}}{192\bar{T}^*\log_2\bar{T}_{r,r}}\ln\frac{1}{\delta_0}\right)$

$$\leq \exp\left(-\frac{T}{768\bar{T}^*\log_2 T}\ln\frac{1}{\delta_0}\right) \qquad (\bar{T}_{r,r} \leq T < 4\bar{T}_{r,r})$$

Therefore, in either case, we have

$$T \ge \bar{T}_{r^*+1,1} \implies \mathbb{P}\left(\tau(\mathcal{M}) \ge T\right) \le \exp\left(-\frac{T}{768\bar{T}^*\log_2 T}\ln\frac{1}{\delta_0}\right)$$

We conclude the proof by working out an explicit upper bound on $\bar{T}_{r^*+1,1}$ as follows: $\bar{T}_{r^*+1,1} \leq \bar{T}_{r^*+1,r^*+1}$

$$\begin{aligned} &+1,1 \leq T_{r^{*}+1,r^{*}+1} \\ &\leq 2^{r^{*}} \cdot 3(r^{*}+1)^{2} L_{1} T_{1} \\ &\leq \frac{8\bar{T}^{*}}{T_{1}} \cdot 3(\log_{2}\frac{16\bar{T}^{*}}{T_{1}})^{2} L_{1} T_{1} \\ &\leq 24\bar{T}^{*} \log_{2}^{2} \left(\frac{16\bar{T}^{*}}{T_{1}}\right) L_{1} \end{aligned}$$
 (Lemma 29)

C.2 Utility Lemmas

Lemma 27. Let
$$\beta > 0$$
, $a \ge 1$, $\delta \in (0,1)$, and $\delta_0 \in (0,1)$. If, $L_1 \ge \alpha \frac{\ln\left(1+\frac{2}{\delta}\right)}{\ln\left(\frac{1}{\beta\delta_0}\right)}$, then

$$\sum_{s=1}^{\infty} \exp\left(-a \cdot \frac{2^{s-1}L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) \le \frac{3}{2} \exp\left(-a \cdot \ln(1+\frac{2}{\delta})\right)$$

Proof.

$$\begin{split} &\sum_{s=1}^{\infty} \exp\left(-a \cdot \frac{2^{s-1}L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) \\ &= \exp\left(-a \cdot \frac{L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) + \exp\left(-a \cdot \frac{2L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) + \exp\left(-a \cdot \frac{4L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) + \cdots \right) \\ &\leq \exp\left(-a \cdot \frac{L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) + \exp\left(-a \cdot \frac{2L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) + \exp\left(-a \cdot \frac{3L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) + \cdots \right) \\ &= \frac{\exp\left(-a \cdot \frac{L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right)}{1 - \exp\left(-a \cdot \frac{L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right)} \end{split}$$
(geometric sum)

Let
$$L_1 \ge \frac{\alpha \ln\left(1+\frac{2}{\delta}\right)}{\ln \frac{1}{\beta\delta_0}}$$
. Then,
 $1 - \exp\left(-a \cdot \frac{L_1}{\alpha} \ln\left(\frac{1}{\beta\delta_0}\right)\right) = 1 - \exp(-a \ln\left(1+\frac{2}{\delta}\right))$
 $= 1 - \frac{1}{\left(1+\frac{2}{\delta}\right)^a}$
 $\ge 1 - \frac{1}{\left(1+\frac{2}{\delta}\right)}$ $(a \ge 1)$
 $\ge 1 - \frac{1}{\left(1+\frac{2}{1}\right)}$ $(\delta \le 1)$

 $=\frac{2}{3}$

Thus,

$$\frac{\exp\left(-a \cdot \frac{L_1}{\alpha} \ln\left(\frac{1}{\beta \delta_0}\right)\right)}{1 - \exp\left(-a \cdot \frac{L_1}{\alpha} \ln\left(\frac{1}{\beta \delta_0}\right)\right)} \leq \frac{3}{2} \exp\left(-a \cdot \ln(1 + \frac{2}{\delta})\right) \ ,$$

which completes the proof.

Lemma 28. Let \mathcal{E} be an event from a random trial such that $\mathbb{P}(\mathcal{E}) \leq \delta$ Let $\alpha \in (\delta, 1)$. Let N be the number of trials where \mathcal{E} holds true out of L independent trials. Then,

$$\mathbb{P}(N \ge \alpha L) \le \exp\left(-\alpha L \log\left(\frac{1}{e\delta}\right)\right)$$

Proof. Recall the standard KL divergence based concentration inequality where $\hat{\mu}_n$ is the sample mean of n Bernoulli i.i.d. random variables with head probability μ :

$$\forall \varepsilon \ge 0, \mathbb{P}(\hat{\mu}_n - \mu \le \varepsilon) \le \exp(-n\mathbf{KL}(\mu + \varepsilon, \mu)) .$$

Note that N/L can be viewed as the sample mean of Bernoulli trials with $\mu := \mathbb{P}(\mathcal{E})$. Then,

$$\begin{split} \mathbb{P}(N \ge \alpha L) &= \mathbb{P}(\frac{N}{L} \ge \alpha) \\ &= \mathbb{P}(\frac{N}{L} - \mu \ge \alpha - \mu) \\ &\le \exp(-L\mathbf{K}\mathbf{L}(\alpha, \mu)) \qquad (\alpha > \delta \ge \mu) \\ &= \exp\left(-L\left(\alpha \ln(\frac{\alpha}{\mu}) + (1 - \alpha) \ln \frac{1 - \alpha}{1 - \mu}\right)\right) \\ &\stackrel{(a)}{\le} \exp\left(-L\left(\alpha \ln(\frac{\alpha}{\mu}) - \alpha\right)\right) \\ &\le \exp\left(-L\left(\alpha \ln(\frac{\alpha}{\mu}) - \alpha\right)\right) \\ &= \exp\left(-L\alpha \ln(\frac{\alpha}{e\mu})\right) \end{split}$$

where (a) is by the following derivation:

$$(1 - \alpha) \ln \frac{1 - \alpha}{1 - \mu} = -(1 - \alpha) \ln \frac{1 - \mu}{1 - \alpha}$$
$$= -(1 - \alpha) \ln \left(1 + \frac{\alpha - \mu}{1 - \alpha} \right)$$
$$(\forall x \quad \ln(1 + x) \le x)$$
$$\geq -(1 - \alpha) \cdot \frac{\alpha - \mu}{1 - \alpha}$$
$$= -(\alpha - \mu)$$
$$\geq -\alpha$$

Hence,

$$\mathbb{P}(N \ge \alpha L) \le \exp\left(-L\alpha \ln(\frac{\alpha}{e\mu})\right) \le \exp\left(-L\alpha \ln(\frac{\alpha}{e\delta})\right) \qquad (\mu = \mathbb{P}(\mathcal{E}) \le \delta)$$

Lemma 29. Let $T_{r,r} = 2^{r-1}T_1$ for some $T_1 \ge 1$ and define $\overline{T}^* := T^* \lor T_1$. Define $r^* := \min\{r \in \mathbb{N}_+ : T_{r,r} \ge T^*\}$. Then,

$$r^* \le \log_2 \frac{8\bar{T}^*}{T_1}$$

Proof. Consider two cases:

(i) $r^* \ge 2$: In this case,

$$T^* > T_{r^* - 1, r^* - 1} = 2^{r^* - 2} T_1$$
$$\implies r^* < \log_2(\frac{4T^*}{T_1})$$

(ii) $r^* \leq 1$: Nothing to do here.

Together, we have $r^* \leq 1 \vee \log_2 \frac{4T^*}{T_1} \leq 1 \vee \log_2 \frac{4\bar{T}^*}{T_1} \leq \log_2 \frac{8\bar{T}^*}{T_1}$ where we use the fact that $\forall a, b \geq 0, a \vee b \leq a + b$ and $\log_2(4\bar{T}^*/T_1) \geq 2 \geq 0$.

Lemma 30. Let $\overline{T}_{r,c}$ be the total number of samples used in Algorithm 2 up to and including stage (r,c). If $r \geq 2, c \in [r]$, then

$$\frac{1}{2} \le \frac{T_{r,c}}{r^2 2^{r-1} T_1 L_1} \le 3 \ .$$

Proof. For the upper bound,

$$\bar{T}_{r,c} \leq \bar{T}_{r,r} \leq \sum_{u=1}^{r} \sum_{c=1}^{r} u^2 2^{u-1} L_1 T_1
= (r^2 2^{r-1} - r 2^{r+1} + 3 \cdot 2^r - 3) L_1 T_1
\leq 2^{r-1} (r^2 + 6) L_1 T_1
\leq 2^{r-1} \cdot 3r^2 L_1 T_1$$
(6 \le 2r^2)

For the lower bound,

$$\begin{split} \bar{T}_{r,c} &= \sum_{u=1}^{r-1} \sum_{c=1}^{u} T_1 L_1 \cdot u \cdot 2^{u-1} + \sum_{v=1}^{c} T_1 L_1 \cdot r \cdot 2^{r-1} \\ &= T_1 L_1 \sum_{u=1}^{r-1} u^2 \cdot 2^{u-1} + T_1 L_1 \cdot c \cdot r \cdot 2^{r-1} \\ &= T_1 L_1 \cdot \left(2^{r-1} \left(r^2 - 4r + 6 \right) - 3 \right) + T_1 L_1 \cdot c \cdot r \cdot 2^{r-1} \\ &\geq T_1 L_1 \cdot \left(2^{r-1} \left(r^2 - 4r + 6 \right) - 3 \right) + T_1 L_1 \cdot r \cdot 2^{r-1} \\ &= T_1 L_1 \cdot \left(2^{r-1} \left(r^2 - 3r + 6 \right) - 3 \right) \\ &= \frac{T_1 L_1}{2} \cdot \left(2^{r-1} \left(2r^2 - 6r + 12 \right) - 6 \right) \\ &= \frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2 + \frac{T_1 L_1}{2} \cdot 2^{r-1} \left(r^2 - 6r + 9 \right) \\ &\geq \frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2 + \frac{T_1 L_1}{2} \cdot 2^{r-1} \left(r^2 - 6r + 9 \right) \\ &= \frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2 + \frac{T_1 L_1}{2} \cdot 2^{r-1} \left(r - 3 \right)^2 \\ &\geq \frac{T_1 L_1}{2} \cdot 2^{r-1} \cdot r^2 + \frac{T_1 L_1}{2} \cdot 2^{r-1} \left(r - 3 \right)^2 \end{split}$$